Curve Sketching

notes made by bearwing

Content

Exam Types

I. Factorized rational function
09, 07, 03, 96

$$f(x)=\frac{(x+15)(x+1)^2}{(x-6)^2}$$

II. Expanded rational function
06, 04, 02, 00, 95

$$f(x)=\frac{x^2-x-6}{x+6}$$

III. Fractional index
05, 01, 98, 97, 94, 93, 91

$$f(x)=x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$$

IV. Exponential function
08, 99, 92

$$f(x)=(2x^2-14x+25)e^{2x}$$

V. Absolute value
09, 05, 04, 03, 95

$$f(x)=\frac{|x|(x+16)}{x-2}$$

Steps for Sketch a Curve

Using $f(x)$ to…

1. Find a correct expression for $f(x)$ if necessary.
2. Find the x,y-intercept for $f(x)$.
3. Find $f'(x)$.

Using $f'(x)$ to…

4. Find $f''(x)$.
5. Construct a table to exam the domain for which
   $f'(x) > 0$ or $f'(x) < 0$
   to determinate the curve is increasing or decreasing.
6. Find the extreme point for $f(x)$.

Using $f''(x)$ to…

7. Construct a table to exam the domain for which
   $f''(x) > 0$ or $f''(x) < 0$
   to determinate the curve is convex or concave.
8. Find the point of inflexion for $f(x)$.

Using $f(x)$ to…

9. Find the asymptote of the graph.
10. Sketch the graph.

Example (HKALE 2003 Q7)	
1	Find a correct expression for $f(x)$. $f(x)=\frac{x\|x+1\|}{x+2}$ $f(x)=\begin{cases} x-1+\frac{2}{x+2} & \text{for } x>-1 \\ -\left(x-1+\frac{2}{x+2}\right) & \text{for } x<-1 \end{cases}$
2	The x-intercept is $\frac{x\|x+1\|}{x+2}=0$ $x\|x+1\|=0$ $x=0$ or $x=-1$ x-intercept is $(0,0)$ and $(-1,0)$. The y-intercept is $f(0)=\frac{0\|0+1\|}{0+2}$ $f(0)=0$ y-intercept is $(0,0)$.
3	Find $f'(x)$. $f'(x)=\begin{cases} 1-\frac{2}{(x+2)^2} & \text{for } x>-1 \\ -\left(1-\frac{2}{(x+2)^2}\right) & \text{for } x<-1 \end{cases}$
4	Find $f''(x)$. $f''(x)=\begin{cases} \frac{4}{(x+2)^3} & \text{for } x>-1 \\ -\left(\frac{4}{(x+2)^3}\right) & \text{for } x<-1 \end{cases}$

5 Construct a table for $f'(x)$.

	$x<-2-\sqrt{2}$	$x=-2-\sqrt{2}$	$-2-\sqrt{2}<x<-2$	$-2<x<-1$
$f'(x)$	−	0	+	+
	↘		↗	↗

	$x=-1$	$-1<x<-2+\sqrt{2}$	$x=-2+\sqrt{2}$	$x>-2+\sqrt{2}$
$f'(x)$	0̸	−	0	+
		↘		↗

6 The extreme point for $f(x)$ are

Minimum points: $(-2-\sqrt{2},3+2\sqrt{2})$ and $(-2+\sqrt{2},2\sqrt{2}-3)$

Maximum points: $(-1,0)$

7 Construct a table for $f''(x)$.

	$x<-2$	$-2<x<-1$	$x=-1$	$x>-1$
$f'(x)$	+	−	0̸	+
	∪	∩		∪

8 The point of inflexion for $f(x)$ is $(-1,0)$

9

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-} -\left(\frac{x(x+1)}{x+2}\right)=+\infty$$

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+} -\left(\frac{x(x+1)}{x+2}\right)=-\infty$$

The vertical asymptote is $x=-2$

$$\lim_{x\to +\infty}\left[f(x)-(x-1)\right]=\lim_{x\to +\infty}\frac{2}{x+2}=0^+$$

$$\lim_{x\to -\infty}\left[f(x)-(1-x)\right]=\lim_{x\to -\infty}\frac{-2}{x+2}=0^+$$

The oblique asymptotes are

$$\begin{cases} y=x-1 & \text{for } x\to +\infty \\ y=1-x & \text{for } x\to -\infty \end{cases}$$

Exam Techniques

Type 1) Differentiation techniques:

1. Chain rule and Quotient rule

Steps	
Chain rule:	Quotient rule:
$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Example (HKALE 2002 Q8)	
$f(x) = x^2 - \frac{8}{x-1}$	
By Chain rule	By Quotient rule
$f(x) = x^2 - 8(x-1)^{-1}$ $\Rightarrow f'(x) = 2x + 8(x-1)^{-2}$	$f(x) = x^2 - \frac{8}{x-1}$ $\Rightarrow f'(x) = 2x + \frac{8}{(x-1)^2}$

Applicable for (5★=max)			
I	II	III	IV
★★	★★★	★	★★★★

Brief introduction:

This method is the fundamental method to differentiate a function. However, sometime using quotient rule will give out a very bulky function and many students may make calculation mistake in this stage. Therefore, using this method is not preferred.

2. Logarithmic function

Steps	
1	Log. the function $y = f(x) \Rightarrow \ln y = \ln f(x)$
2	Differentiate the log. function $\ln y = \ln f(x)$ $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln f(x)$ $\frac{dy}{dx} = f(x) \bullet \frac{d}{dx}\ln f(x)$

Example (HKALE 2007 Q7)	
$f(x) = \frac{(x+15)(x+1)^2}{(x-6)^2}$	
1	$\ln f(x) = \ln(x+15) + 2\ln(x+1) - 2\ln(x-6)$
2	$\frac{1}{f(x)} \bullet f'(x) = \frac{1}{x+15} + \frac{2}{x+1} - \frac{2}{x-6}$ $f'(x) = \left(\frac{(x+15)(x+1)^2}{(x-6)^2}\right)\left(\frac{(x+1)(x-6)+2(x+15)(x-6)-2(x+15)(x+1)}{(x+15)(x+1)(x-6)}\right)$ $= \left(\frac{(x+1)}{(x-6)^3}\right)\left(x^2-5x-6+2x^2+18x-180-2x^2-32x-30\right)$ $= \frac{(x+1)(x^2-19x-216)}{(x-6)^3}$ $= \frac{(x+1)(x+8)(x-27)}{(x-6)^3}$

Applicable for (5★=max)			
I	II	III	IV
★★★★	★★★	★★★★★	★★★★

Brief introduction:

This method is applicable for most of the cases. It is quite convenient and quick. Usually only polynomial up to degree2 is needed for calculation. The only disadvantage of this method is that, sometimes using this method is mathematically incorrect. This is because the function inside log. must be positive. When you take natural log. on both side of the function, you may conduct an error . Although the result obtained is still correct, but in mathematically meaning, it is wrong in deed.

Something you should note:

1. Don't forget to differentiate $-x$.

   $$\frac{d}{dx}\ln(a-x)=\frac{-1}{a-x}$$

2. Don't forget to differentiate function inside log.

   $$\frac{d}{dx}\ln(x^2-x+a)=\frac{2x-1}{x^2-x+a}$$

3. Long division

Steps	
1	Carry out long division on the function $f(x)=\dfrac{p(x)}{q(x)}=a(x)+\dfrac{k}{q(x)}$, for some constant k.
2	Differentiate the function $f(x)=a(x)+\dfrac{k}{q(x)}$ $f'(x)=a'(x)-k\dfrac{q'(x)}{q(x)^2}$

Example (HKALE 2006 Q7)	
$f(x)=\dfrac{x^2-x-6}{x+6}$	
1	$f(x)=\dfrac{x^2-x-6}{x+6}$ $=x-7+\dfrac{36}{x+6}$
2	$f'(x)=1-\dfrac{36}{(x+6)^2}$ $=\dfrac{x(x+12)}{(x+6)^2}$

Applicable for (5★=max)			
I	II	III	IV
★★★★★	★★★★★	／	／

Brief introduction:

This method is the best method to differentiate rational functions. It can help you to find the first and second derivatives of the function easily, and further facilitate the process for finding oblique asymptote.

4. Exponentiation

Steps	
1	Power the function $f(x) = g(x)^{\frac{1}{n}} \Rightarrow f(x)^n = g(x)$
2	$nf(x)^{n-1} f'(x) = g'(x)$ $f'(x) = \dfrac{g'(x)}{nf(x)^{n-1}}$

Example (HKALE 1993 Q8)	
$f(x) = \sqrt[3]{x^2 - x^3}$	
1	$f(x)^3 = x^2 - x^3$
2	$3f(x)^2 f'(x) = 2x - 3x^2$ $f'(x) = \dfrac{x(2-3x)}{3(x^2 - x^3)^{\frac{2}{3}}}$

Applicable for (5★=max)			
I	II	III	IV
／	／	★★★★	／

Brief introduction:

This method is restricted to some special type of fractional index question. Therefore, its use is limited to a few case only.

Type 2) Asymptote techniques:

1. Direct checking

Steps	
Take limit on the point that will cause the function to infinity. If $\lim_{x\to c} f(x) = \infty$ Then $x = c$ is a vertical asymptote.	Take limit on infinity. If $\lim_{x\to\infty} f(x) = k$ for some constant k Then $y = k$ is a horizontal asymptote.

Example (HKALE 2002 Q8 and 2000 Q9)	
$f(x) = x^2 - \frac{8}{x-1}$	$f(x) = \frac{x}{(1+x^2)^2}$
$\lim_{x\to 1} \frac{8}{x-1} \to \infty$ $\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(x^2 - \frac{8}{x-1}\right) \to \infty$ Hence, $x = 1$ is a vertical asymptote for $f(x)$.	$\lim_{x\to\infty} \frac{x}{(1+x^2)^2} = \lim_{x\to\infty} \frac{1}{4x(1+x^2)} = 0$ Hence, $y = 0$ is a horizontal asymptote for $f(x)$ as x approach $\pm\infty$.

Brief introduction:

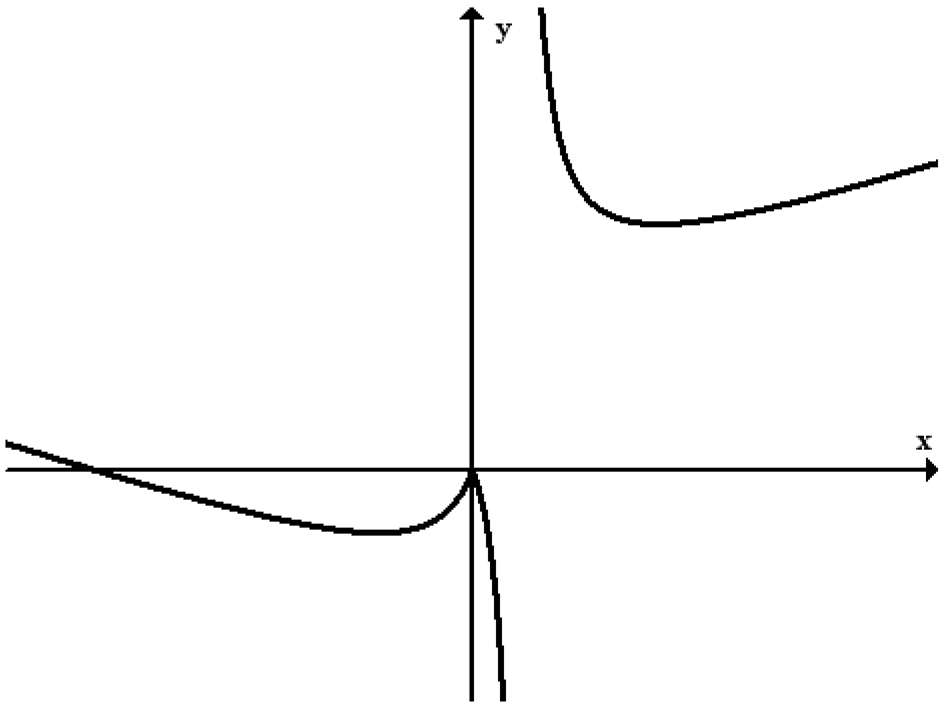
This method is 100% useful for finding vertical asymptote and horizontal asymptote. Almost all function will have vertical asymptote so do not forget to use it as you can at least obtain 1 mark.

Something you should note:

1. Pay attention to horizontal asymptote and vertical asymptote when the type of question is exponential function.
   (Usually, they do not have oblique asymptote.)

2. Horizontal asymptote will appear if
   i) The function is multiplied by a exponential function like e^x.
   ii) The numerator is smaller than denominator.

$$f(x)=\frac{x}{(1+x^2)^2}$$



2. Traditional method

Steps	
1	Find $m=\lim_{x\to\infty}\frac{f(x)}{x}$
2	Find $b=\lim_{x\to\infty}\left[f(x)-mx\right]$
3	$y=mx+b$ is the asymptote if m and b both exist.

Example (HKALE 1996 Q8)	
$f(x)=\frac{(x-1)^3}{(x+1)^2}$	
1	$m=\lim_{x\to\infty}\frac{(x-1)^3}{x(x+1)^2}=\lim_{x\to\infty}\frac{(1-\frac{1}{x})^3}{(1+\frac{1}{x})^2}=1$
2	$b=\lim_{x\to\infty}\left[\frac{(x-1)^3}{(x+1)^2}-x\right]$ $=\lim_{x\to\infty}\frac{(x-1)^3-x(x+1)^2}{(x+1)^2}$ $=\lim_{x\to\infty}\frac{x^3-3x^2+3x-1-x^3-2x^2-x}{(x+1)^2}$ $=\lim_{x\to\infty}\frac{-5x^2+2x-1}{(x+1)^2}$ $=\lim_{x\to\infty}\frac{-5+\frac{2}{x}-\frac{1}{x^2}}{(1+\frac{1}{x})^2}$ $=-5$ $y=x-5$ is the oblique asymptote for $f(x)$.

Brief introduction:

This method is applicable for all function which has oblique asymptote. It also helps you to test whether a function has oblique asymptote or not.

Something you should note:

1. If you find that m or b not exists, then the function does not have oblique asymptote.

$$f(x)=x^2-\frac{8}{x-1}$$

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left[x-\frac{8}{x(x-1)}\right]\to\infty$$

Hence, there is no oblique asymptote.

2. If you have used long division to differentiate function, then you may skip step 1,2 by directly doing the following:

$$f(x)=\frac{x^2-x-6}{x+6}$$

$$=x-7+\frac{36}{x+6}$$

i.e. $\lim_{x\to\infty}\left[f(x)-(x-7)\right]=\lim_{x\to\infty}\frac{36}{x+6}=0$

Hence, $y=x-7$ is the oblique asymptote for $f(x)$.

3. Sometimes, you may use L' Hospital's rule to find m and b.

$$f(x)=x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$$

$$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left(\frac{6}{x}-1\right)^{\frac{1}{3}}=-1$$

$$b=\lim_{x\to\infty}\left[f(x)-(-x)\right]=\lim_{x\to\infty}[x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}+x]$$

$$=\lim_{x\to\infty}\frac{\left(\frac{6}{x}-1\right)^{\frac{1}{3}}+1}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}2\left(\frac{6}{x}-1\right)^{-\frac{2}{3}}\bullet\frac{\frac{d}{dx}\frac{1}{x}}{\frac{d}{dx}\frac{1}{x}}$$

$$=2$$

4. If the function is in the form of $f(x)=\frac{p(x)}{q(x)}$ and

$$\begin{cases} p(x)=ax^{p}+a_1x^{p-1}+\dots \\ q(x)=bx^{q}+b_1x^{q-1}+\dots \end{cases}$$

Then, the value of m can be approximate by

$$m=\lim_{x\to\infty}\frac{ax^{p}}{bx^{q}}$$

Hence,

i) If $p>q$, then $m\to\infty$.

ii) If $p=q$, then $m=\frac{a}{b}$.

iii) If $p<q$, then $m=0$.

Absolute Value

1. If the function contains absolute value, divide the function into two parts by considering the value for which x will cause the function inside absolute sign to zero.

$$f(x)=\frac{x|x+1|}{x+2}$$

$$f(x)=\begin{cases} x-1+\dfrac{2}{x+2} & \text{for } x>-1 \\ -\left(x-1+\dfrac{2}{x+2}\right) & \text{for } x<-1 \end{cases}$$

$$f'(x)=\begin{cases} 1-\dfrac{2}{(x+2)^2} & \text{for } x>-1 \\ -\left(1-\dfrac{2}{(x+2)^2}\right) & \text{for } x<-1 \end{cases}$$

2. While checking range of $f'(x)$ and $f''(x)$, just use the positive part to check and do not consider the negative part even x is not in the range of positive part. Then, reverse the sign of the result obtained if x is not in the range of positive part.

	$x<-2-\sqrt{2}$	$x=-2-\sqrt{2}$	$-2-\sqrt{2}<x<-2$	$-2<x<-1$
$f'(x)$	+	0	–	–

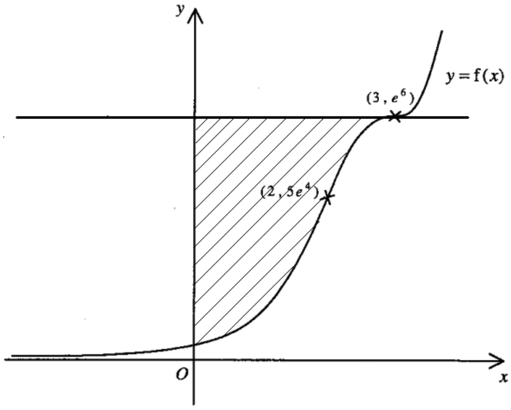
↓

	$x<-2-\sqrt{2}$	$x=-2-\sqrt{2}$	$-2-\sqrt{2}<x<-2$	$-2<x<-1$
$f'(x)$	–	0	+	+

Last part Analysis

1. Integration (08, 06)

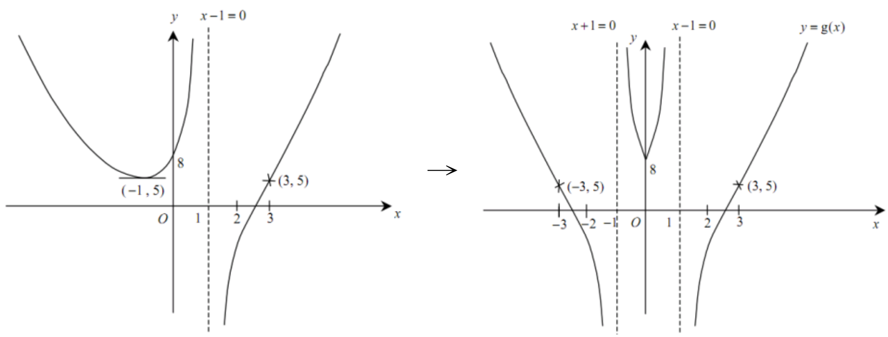
Steps	
1	Sketch the graph and shade the required area.
2	Find the area by integration.

<table>
<tr><td colspan="2">Example (HKALE 08 Q7)</td></tr>
<tr><td colspan="2">$f(x)=\left(2x^2-14x+25\right)e^{2x}$</td></tr>
<tr><td>1</td><td>
</td></tr>
<tr><td>2</td><td>Since $\int_0^3 f(x)dx=\frac{3}{2}e^6-\frac{33}{2}$,
The required area
$=3e^6-\int_0^3 f(x)dx$
$=\frac{3}{2}e^6+\frac{33}{2}$</td></tr>
</table>

2. Sketch Absolute Function (02, 00, 96)

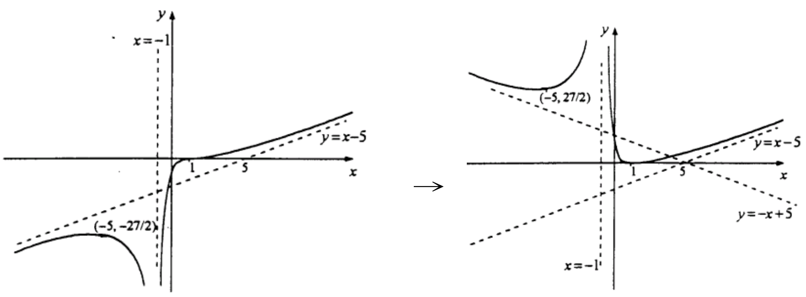
a) Sketch $f(|x|)$.

Note that $f(|-x|) = f(|x|)$, so $f(|x|)$ is an even function.
When doing this type of question, ignore the part which $x < 0$ and only use the part which $x > 0$ on the curve.



b) Sketch $|f(x)|$.

When doing this type of question, reflect the curve about x-axis for which $f(x) < 0$.



3. Rotation (92)

(This type of question is skipped as it has not been seen for a very long time.)

Remarks

1. If the function is even function, then it is symmetrical about y-axis. If the function is odd function, then reflect the curve for $x > 0$ by x-axis and y-axis.

2. While constructing a table, you should divide the table for which x will cause:
   a) $f'(x) = 0$ or $f''(x) = 0$
   b) $f(x) = \infty$ or $f'(x) = \infty$ or $f''(x) = \infty$
   c) The function inside absolute sign $= 0$
   (if the question contains absolute value)

3. If the question contains absolute value or fractional index, usually it will also ask you whether some points are differentiable or not.

Example (HKALE 2003 Q7)
$f(x) = \dfrac{x\|x+1\|}{x+2}$
$\lim_{h\to 0^-} \dfrac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^-} \dfrac{(-1+h)\|h\|}{h(h+1)} = \lim_{h\to 0^-} \dfrac{(-1+h)(-h)}{h(h+1)} = 1$ $\lim_{h\to 0^+} \dfrac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^+} \dfrac{(-1+h)\|h\|}{h(h+1)} = \lim_{h\to 0^+} \dfrac{(-1+h)(h)}{h(h+1)} = -1$ As $f'_-(-1) \neq f'_+(-1)$, f is not differentiable at -1.

4. Note that even if $f'(a)$ and $f''(a)$ not exist for some constant a, if f is continuous on $x=a$, then the relative extreme points and points of inflexion may still exist.

Example (HKALE 2003 Q7)
$f(x)=\frac{x\|x+1\|}{x+2}$
Both $f'(-1)$ and $f''(-1)$ doesn't exist but $f(x)$ is continuous on $x=-1$.

	$-2<x<-1$	$x=-1$	$-1<x<-2+\sqrt{2}$
$f'(x)$	$+$	~~0~~	$-$

	$-2<x<-1$	$x=-1$	$x>-1$
$f''(x)$	$-$	~~0~~	$+$

Hence, $(-1,0)$ is the maximum point and point of inflexion of the function $f(x)=\frac{x|x+1|}{x+2}$.

5. Be very careful about the exponential functions. They may have different limits when approaching 0^+ and 0^-.

Example (HKALE 1999 Q8)
$f(x)=xe^{\frac{1}{x}}$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} xe^{\frac{1}{x}}=0$ because $e^{-\infty}\to 0$.

But $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{e^{\frac{1}{x}}}{\frac{1}{x}}=\lim_{x\to 0^+} e^{\frac{1}{x}}\to +\infty$.

$\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$!!!

6. Input the following program in your calculator so you can avoid making mistake while checking the range of $f'(x)$ and $f''(x)$. Be careful that your calculator should be in **radian mode**.

Casio FX-3650P / 3950P / Truly SC-185

	at least 84 bytes
1	Mem clear : ? → X : 500 → D :
2	Lbl 1 : (-1)^MM ÷ D + X → X : 1 M+ :
3	**X - cosX**
4	→ C : M = 1 => C → A :
5	M = 2 => C → B : 3 > M => Goto 1 :
6	B → X : A ◢ 2^{-1}D(C – B → B ◢
7	D^2(C + X – 2A → C

Casio FX-50FH / 50F Plus

	at least 84 bytes
1	ClrMemory : ? → X : 500 → D : While M < 3 :
2	(-1)^(M) M ÷ D + X → X : 1 M+ :
3	**X - cosX**
4	→ C : M = 1 => C → A : M = 2 => C → B :
5	WhileEnd : B → X : A ◢ 2^{-1}D(C – B → B ◢
6	D^2(C + X – 2A → C

(The M+ should be one button on the calculator, but not M and +)

(You should change the function **X – cosX** *into the function you want)*

Example
$f(x) = x - \cos x$
To find $f(0.5)$, $f'(0.5)$ and $f''(0.5)$, In put D = 0.5 into the program, $f(0.5) = -0.37758$ $f'(0.5) = 1.47943$ $f''(0.5) = 0.87758$