Comparison between *Pure Mathematics Curriculum and Assessment Guide (Advanced Level)* and *Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992*

This Pure Mathematics Curriculum and Assessment Guide (Advanced Level) is a revised edition of the version published in 1992. Some topics have been deleted or trimmed. For ease of reference of teachers, these topics are enclosed in boxes like ▒▒▒ from the 1992 version, page for page. Notes and additional remarks are enclosed in boxes like $\boxed{}$ to delimit the complexity of teaching.

3. SYLLABUS

UNIT A1: The Language of Mathematics

Objective: (1) To understand the first notion of set language.

(2) To understand the first notion of logic.

Detailed Content	Time Ratio	Notes on Teaching
1.1 Set language	5	Basic terminology to be introduced includes set, element, subset, mother set, power set, empty (void) set, equal sets, disjoint sets, universal set, intersection, union, complement, and product of sets. It should be noted that in ushering in the foregoing concepts, just informal treatment is expected and teachers are encouraged to adopt an adequately wide spectrum of simple and factual examples of daily life nature to support their teaching. Conventionally used symbols and notations should be clearly taught. The following are for reference. (1) Sets are generally denoted by capital letters and elements by small letters. The sets of numbers listed below are commonly denoted by the accompanying symbols: the set of all natural numbers **N** the set of all integers **Z; I** the set of all rational numbers **Q** the set of all real numbers **R** the set of all complex numbers **C** (2) Sets are usually presented either in tabular form i.e. with all the elements listed out like A = { 2, 4, 6, 8, 10 } or in propositional form { x:p (x) } like A = { x:x $\le$ 10, x is an even and positive integer } (3) Just simple and straight forward operation rules on intersection, union and complement may be introduced to substantiate students' learning. It is advisable to use Venn diagrams to offer intuitive understanding of the rules as it is probably the first time for the students to come across terms like commutative, associative and distributive etc.

Detailed Content	Time Ratio	Notes on Teaching
1.2 Simple logic	5	Basic terminology to be introduced includes statement/proposition, truth value, conjunction, disjunction, negation, conditional and biconditional, equivalent statements, equivalence, implication, quantifiers, examples and counter-examples. The use of truth table to manifest the meaning of the above connectives is an advisable approach. It is anticipated that more emphasis will be directed to the teaching of conditional and biconditional in the form of "if-then" and "if-and-only-if" which are very widely used in the study of mathematics. Teachers should also touch upon "theorem" and "converse". In teaching this topic, teachers should provide adequate relevant daily life statements for illustration in the first place and then students should be encouraged to give examples of their own. It must be noted that class discussion with the students is helpful in bringing around the concept and using it as a tool. The pure analytical approach is not desirable. To reinforce students' understanding regarding 'sufficient condition', 'necessary condition' and 'necessary and sufficient condition', teachers may lead a discussion with them using simple examples as follows which consist of two propositions and teachers may proceed to investigate with students which condition(s) is (are) applicable: (1) x and y are integers; xy is an integer. (2) x and y are even; x+y is even. (3) x and y are even; x+y and xy are even. (4) The equation $ax^2 + bx + c = 0$ has equal roots; $b^2 - 4ac = 0$. Furthermore, the fact that $(p \to q) \equiv (\sim q \to \sim p)$ should be elaborated (commonly known as contrapositive) with reference to some simple results like the above-mentioned examples. Also teachers may demonstrate some proofs using the method of contradiction. For example, to prove $\sqrt{2}$ is irrational is a commonplace.
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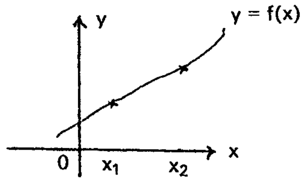
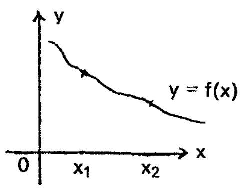
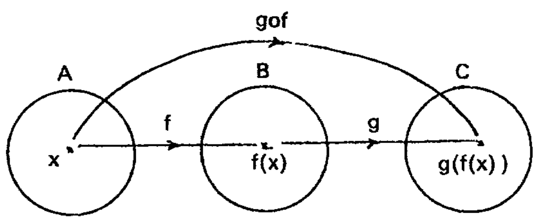
Unit A2: Functions

Objective : (1) To recognize function as a fundamental tool in other branches of mathematics.

(2) To sketch and to describe the shapes of different functions.

Detailed Content	Time Ratio	Notes on Teaching
2.1 Functions and their graphs	2	Students should be taught with a clear definition of function, however rigorous treatment is not expected. The following version may be adopted: $f: A \to B$, f is a function from A to B if every element in A associated with an unique element in B. A is called the domain of f, B the range of f. For an element x in A, the element in B which is associated with x under f usually denoted by f(x) is called the image of x under f and $f[A]$ denotes the set of images of A under f. Special emphasis should be put on real-valued functions since they are most useful in the discussion of other mathematical topics in this syllabus. Students are also expected to be able to plot/ sketch the graph of functions.
2.2 Properties and operations of functions	4	Clear definitions of injective, surjective and bijective functions should be taught so that students are able to distinguish them and to apply the knowledge to solve problems concerned. The following suggested versions may be adopted: A function $f: A \to B$ is (i) injective (one-to-one) if and only if for elements a_1, a_2 in A, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$, or equivalently, $f(a_1) = f(a_2)$ implies $a_1 = a_2$; (ii) surjective (onto) if and only if $f[A] = B$ i.e. every element in B is the image of an element in A; (iii) bijective (one-to-one correspondence) if and only if f is injective and surjective. At this juncture, teachers may provide sufficient preparation on the part of the students so that the concept of inverse function denoted by f^{-1} can be easily figured out and the property that f is bijective if and only if its inverse function f^{-1} exists. Moreover the property that the graphs of a function and its inverse (if exists) are reflections about the line y = x should be studied with adequate illustrations. Students should be able to distinguish odd, even, periodic, increasing and decreasing 'functions. In this connection, clear definitions should be provided. For increasing and decreasing functions, prior to the teaching of differential calculus, teachers may adopt the following suggested .definitions: (i) f(x) is increasing (strictly increasing) if and only if $x_2 > x_1$ implies $f(x_2) \geq f(x_1)$ $(f(x_2) > f(x_1))$;

Detailed Content	Time Ratio	Notes on Teaching
		(ii) f(x) is decreasing (strictly decreasing) if and only if $x_2 > x_1$ implies $f(x_2) \leq f(x_1)$ $(f(x_2) < f(x_1))$.  f(x) is strictly increasing  f(x) is strictly decreasing These properties may be useful in sketching curves and in evaluating definite integrals, etc. Concerning the operations with functions, teachers should discuss with the students that, for functions f and g, f + g, f – g, f × g and $\frac{f}{g}$ (provided $g(x) \neq 0$ for all values of x concerned) are again functions. However regarding the composition of functions which is very useful especially in teaching the chain rule in differential calculus, it is desirable to give ample simple examples so as to support students' mastery of the concepts. Teachers may consider the following suggestion with due emphasis directed to real-valued functions: If $f: A \to B$ and $g: B \to C$ are functions, then the composition of f and g is the function $g \circ f: A \to C$ such that $g \circ f(x) = g(f(x))$ for all element x in A. 

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Detailed Content	Time Ratio	Notes on Teaching
2.3 Algebraic functions	2	Students should be able to recognize the- following algebraic functions: *(a)* polynomial functions *(b)* rational functions *(c)* power functions x^{α} where α is rational *(d)* other algebraic functions derived from the above-mentioned ones through addition, subtraction, multiplication, division and composition like $\sqrt{x^2+1}$
2.4 Trigonometric functions and their formulae	14	Students should be able to sketch the graphs of the six trigonometric functions and their inverse functions. The basic relations like $\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ should also be included in the discussion with the students. Simplification of these functions at $(\frac{n\pi}{2} \pm \theta)$ for odd and even n and proving of identities are expected. The knowledge and related applications of the following are also expected: (1) compound angle formulae $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (2) Multiple angle formulae $\sin 2A = 2 \sin A \cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $= 2\cos^2 A - 1$ $= 1 - 2\sin^2 A$ $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ $\sin 3A = 3\sin A - 4\sin^3 A$ $\cos 3A = 4\cos^3 A - 3\cos A$ $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

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Detailed Content	Time Ratio	Notes on Teaching
		(3) Half-angle formulae $\sin^2 \frac{A}{2} = \frac{1}{2}(1-\cos A)$ $\cos^2 \frac{A}{2} = \frac{1}{2}(1+\cos A)$ $\sin A = \frac{2t}{1+t^2}$, $\cos A = \frac{1-t^2}{1+t^2}$ with $t \equiv \tan\frac{A}{2}$ (4) sum and product formulae $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ $\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$ $\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$ $\cos A \sin B = \frac{1}{2}(\sin(A+B)-\sin(A-B))$ $\cos A \cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$ $\sin A \sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$ As a matter of fact, most of the formulae are direct consequence from the basic ones thus teachers should encourage their students to work out the proofs for

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Detailed Content	Time Ratio	Notes on Teaching
		themselves. The transformation of the expression acosx + bsinx into the form rsin(x + α) or rcos(x + β) should be discussed. Applications of the above formulae in proving identities and in solving trigonometric equation should be included.
2.5 Exponential and logarithmic functions	6	Students should know the relation between the exponential and logarithmic functions, viz. one is the inverse function of the other. The common definition of logarithm, like $\log_a x = y$ if and only if $x = a^y$ with a > 0 and a $\neq$ 1 should be revised. As for logarithmic functions of a variable x, students should know that they are functions of logx or of the logarithm of some function of x, like $(\log_{10} x)^2$ and $\log_e(1+\tan x)$ etc. Some common properties of the logarithmic function should be studied: For $f(x) = \log_a x$ with a > o and a $\neq$ 1; (i) $f(x)$ is defined for x > 0 only; (ii) $f(x)$ is an increasing function if a > 1 and is a decreasing function if 0 < a < 1; (iii) for b, c > 0 and b $\neq$ 1, $\log_a c = \dfrac{\log_b c}{\log_b a}$; (iv) $f(a) = \log_a a = 1$; (v) $f(1) = \log_a 1 = 0$. For the exponential function, a parallel treatment should be provided, viz, a function of the form a^x where a is a positive constant and x a variable is called an exponential function. Those common properties of the exponential function include the following: For $f(x) = a^x$ with a > 0 and a $\neq$ 1, (i) f(x) is defined for all real x; (ii) (x) is an increasing function if a > 1 and is a decreasing one if 0 < a < 1; (iii) $f(0) = a^0 = 1$. Students should be able to sketch the graph for (i) the logarithmic function $f(x) = \log_a x$ for the cases a > 1 and 0 < a < 1 (ii) the exponential function also for the cases a > 1 and 0 < a < 1.

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Detailed Content	Time Ratio	Notes on Teaching
		At this juncture, teachers may embark on the following important results so as to extend students' perspective on logarithmic and exponential functions (i) $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$ (ii) $\lim\limits_{h\to 0}(1+h)^{\frac{1}{h}} = e$ and (iii) $\log_e x = \int_1^x \frac{1}{t}\,dt$. (Please note that the third one is optional which could be taken as an alternative definition for $\log_e x$ or written as $\ell n\, x$). The general functional properties of logarithmic and exponential functions should also be mentioned. Logarithmic function $f(x) = \log_a x$ with a > 0 , a $\neq$1. (i) $f(x) + f(y) = f(xy)$ (ii) $f(x) - f(y) = f(\frac{x}{y})$ (iii) $f(x^n) = nf(x)$ and Exponential function $g(x) = a^x$ with a > 0 (i) $g(x+y) = g(x)\cdot g(y)$ (ii) $g(x-y) = \dfrac{g(x)}{g(y)}$ (iii) $g(nx) = (g(x))^n$ And, in particular, the importance of the functions e^x and $\ell n\, x$ in the study of mathematics should be emphasized.
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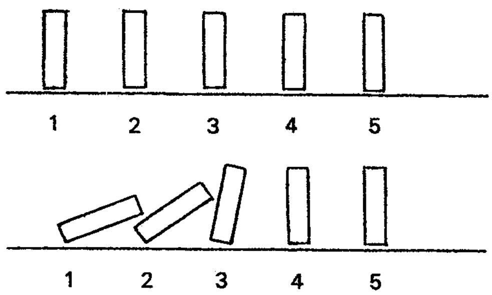
Unit A3: Mathematical Induction

Objective: (1) To understand the Principle of Mathematical Induction.
(2) To apply the Principle of Mathematical Induction to prove propositions involving integers.
(3) To be able to modify the Principle of Mathematical Induction to suit different purposes.

Detailed Content	Time Ratio	Notes on Teaching
3.1 The Principle of Mathematical Induction and its applications	6	As an introduction, students may be asked to guess the formula for the sum of the first n odd positive integers by considering 1 = 1 1 + 3 = 4 1 + 3 + 5 = 9 …………….. …………….. After the proposition $1 + 3 + 5 + . . . + (2n - 1) = n^2$ is established, students should be led to understand that they should not claim this result is true by considering only a finite number of cases. An illustration of the use of mathematical induction should then follow. The Principle of Mathematical Induction should be formally written on the board. Teachers may find it easier to explain the Principle by referring to a game of dominoes:  Examples should be done on the applications to the summation of series, divisibility and proving inequalities. The Principle of Mathematical Induction may

Detailed Content	Time Ratio	Notes on Teaching
		sometimes modified to suit different cases. Examples should also be used to illustrate that both conditions of the Principle must be satisfied to prove a proposition. Further applications include the proofs of (i) the binomial theorem for positive integral indices (ii) De Moivre's theorem for a positive integer n (iii) some propositions involving determinants and square matrices (iv) Leibniz's Theorem and some propositions involving the nth derivative.
3.2 Other common variations of the Principle of Mathematical Induction and their applications $\boxed{\text{Excluding Backward Induction}}$	~~6~~ 5	As further development, teachers may discuss with the students cases where the Principle has to be modified. *Example:* $x^n + y^n$ is divisible by x + y for all positive odd integers n. *Example:* The Fibonacci sequence is defined as follows: $a_0 = 0, a_1 = 1$ $a_{n+1} = a_{n-1} + a_n$ for all natural numbers n. Prove that $a_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$ for all n Teachers should point out that a variation of the Principle is required for the proof of these examples. A few more examples on sequences defined by recurrence relations may be discussed.
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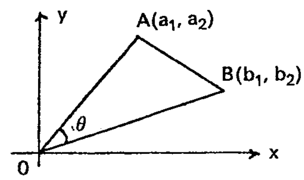
Unit A4: inequalities

Objectives: (1) To learn the elementary properties of inequalities.

(2) To prove simple absolute inequalities.

(3) To solve simple conditional inequalities.

Detailed Content	Time Ratio	Notes on Teaching
4.1 Absolute inequalities	6	Students are expected to use the symbols a > b and a ≥ b correctly. Elementary properties of inequalities including (i) For any real number x, $x^2 \geq 0$ (ii) a > b > 0 and n is a positive integer, then $a^n > b^n$ and $\sqrt[n]{a} > \sqrt[n]{b}$ (iii) If a > b > 0 and x > y > 0, then ax > by should be revised. Formal proofs of these properties are not required. However, students are expected to be able to deduce simple absolute inequalities from the elementary properties. The following techniques in proving absolute inequalities should be emphasized: *Example*: Prove that $E_1 \geq E_2$ Proof: $E_1 - E_2 = ...$ $= ......$ $= ......$ ≥ 0 $\therefore E_1 \geq E_2$
4.2 A.M. ≥ G.M.	4	The proof of A.M. ≥ G.M. may be provided up to four variables in the first instance and the general proof need not be emphasized. If required, teachers may apply backward induction. Students are expected to apply this result to n variables.
4.3 Cauchy—Schwarz's inequality	3	Students are expected to understand that the necessary and sufficient conditions for the quadratic form $ax^2 + bx + c$ to be positive for all real values of x are a > 0 and $b^2 - 4ac < 0$. Students should be able to apply this result to problems such as finding the range of C for which the expression $Cx^2 + 4x + C + 3$ is positive for all values of x. The above result may be used to prove the Cauchy–Schwarz's inequality.

Detailed Content	Time Ratio	Notes on Teaching
		A geometric interpretation of the Cauchy—Schwarz's inequality for n =2 may be given by using points on the coordinate plane. $\cos\theta = \dfrac{OA^2 + OB^2 - AB^2}{2OA \cdot OB}$ $= \dfrac{(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - [(a_1 - b_1)^2 + (a_2 - b_2)^2]}{2\sqrt{(a_1^2 + a_2^2)}\sqrt{(b_1^2 + b_2^2)}}$ $= \dfrac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$ $\cos^2\theta = \dfrac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}$  Since $\cos^2\theta \leq 1$, $(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ Students are also expected to apply the Cauchy—Schwarz's inequality in solving simple problems.
4.4 Conditional inequalities	7	The concepts of intervals on the real number line should be revised. The definition and properties of the absolute value of a real number should be discussed. Students should be able to solve linear inequalities, quadratic inequalities and inequalities of higher degrees in x. Solutions of inequalities involving absolute values such as $\lvert ax^2 + bx + c \rvert \geq d$, $\lvert x - a \rvert + \lvert x - b \rvert \geq c$ and $(x - a)\lvert x - b \rvert \geq c$ are required. Teachers should also discuss with students inequalities of the form $\dfrac{P(x)}{Q(x)} \geq 0$, where P(x) and Q(x) are polynomials in x. Compound inequalities of the above inequalities should also be taught.
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Unit A5: The Binomial Theorem for Positive Integral Indices

Objective: (1) To learn and apply the binomial theorem for positive integral indices.
(2) To study the simple properties of the binomial coefficients.

Detailed Content	Time Ratio	Notes on Teaching
5.1 The binomial theorem for positive integral indices	3	Students should learn how to evaluate n! and C_r^n. The binomial theorem for positive integral indices may be proved by the Principle of Mathematical Induction. Discussions concerning the notation C_r^n should be related to its use as a binomial coefficient. The Pascal's triangle in relation to the coefficients C_r^n in the binomial expansion may be discussed. Students are not expected to know the general binomial theorem.
5.2 Application of the binomial theorem for positive integral indices	5	Students should be able to expand expressions using the binomial theorem for positive integral indices. The determination of a particular term or a particular coefficient in a binomial expansion should also be taught. Students are expected to be able to find the greatest term and the greatest coefficient in a binomial expansion. Applications to numerical approximation should be discussed.
5.3 Simple properties of the binomial coefficients	5	Students should know that both the notations C_r^n and $\binom{n}{r}$ may be used to represent the binomial coefficients. Discussions should include simple properties of the binomial coefficients and the relations between these coefficients such as $C_0^n + C_1^n + C_2^n + \ldots + C_n^n = 2^n$; $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}$ and similar relations. N.B. Permutation and combination may be used to introduce the binomial theorem but problems concerning permutation and combination are not required. Problems involving the use of differentiation and integration may be taught after students have learnt calculus.
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Unit A6: Polynomials and Equations

Objective: (1) To learn the properties of polynomials with real coefficients in one variable.
(2) To learn division algorithm, remainder theorem and Euclidean algorithm and their applications.
(3) To resolve rational functions into partial fractions.
(4) To learn the properties of roots of polynomial equations with real coefficients in one variable.

Detailed Content	Time Ratio	Notes on Teaching
6.1 Polynomials with real coefficients in one variable	5	Students are expected to know the general form of a polynomial with real coefficients in one variable and the following terms: degree of a non-zero polynomial, leading coefficient, constant term, monic polynomial, null or zero polynomial. Definitions of equality, sum, difference and product of two polynomials should also be studied. Also from definition, it is clear that for non-zero polynomials f(x), g(x) deg { f(x) g(x) } = deg f(x) + deg g(x). and deg {f(x) + g(x)} $\leq$ max {deg f(x), deg g(x)} The greatest common divisor (G.C.D.) or highest common factor (H.C.F.) of two non-zero polynomials should be defined. Students should clearly distinguish between division algorithm and Euclidean algorithm. By the division algorithm, the remainder theorem can be proved. Since students have studied the remainder theorem in lower forms, more difficult problems on this theorem can -be given. The Euclidean algorithm is a method of finding the G.C.D. of two polynomials. Some problems on finding the G.C.D. of two polynomials should be given as exercise.
6.2 Rational functions	4	A rational function should be defined first. Students may come across partial fractions the first time. Teacher may quote a simple example such as $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$. The fractions $\frac{1}{x}$ and $\frac{1}{x+1}$ are called partial fractions. The rules for resolving a proper rational function into partial fractions should be clearly stated and examples studied. It should be emphasized and illustrated by examples that if a given rational function is improper, it should first be expressed as the sum of a polynomial and a proper fraction. Applications of partial fractions should be studied.

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Detailed Content	Time Ratio	Notes on Teaching
		Examples: 1. Express $\dfrac{x^2-8x+9}{(x+1)(x-2)^3}$ in partial fractions. 2. Resolve $\dfrac{x^4}{(x-a)(x^2+a^2)}$ into partial fractions. 3. Evaluate $\sum_{r=1}^{n}\dfrac{1}{r(r+1)}$
6.3 Polynomial equations with real coefficients in one variable	6	For the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ with α, β as roots, students should be familiar with the relations $\alpha+\beta=-\dfrac{b}{a}$, $\alpha\beta=\dfrac{c}{a}$ For the general polynomial equation of degree n, $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ or $\sum_{k=0}^{n}a_kx^k=0$, the following theorem gives the relations between coefficients and roots: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the polynomial equation $f(x) = 0$, then the sum s_k of all possible products of the α_j's taken k at a time (k = 1 , 2, . . . , n) is equal to $(-1)^k\dfrac{a_{n-k}}{a_n}$ i.e. if $f(x)=a_n(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$, then $s_1=\alpha_1+\alpha_2+\ldots+\alpha_n=-\dfrac{a_{n-1}}{a_n}$, $s_2=\alpha_1\alpha_2+\alpha_2\alpha_3+\ldots+\alpha_{n-1}\alpha_n=\dfrac{a_{n-2}}{a_n}$, $s_3=\sum\alpha_1\alpha_2\alpha_3=-\dfrac{a_{n-3}}{a_n}$, ………………………… ………………………….. $s_n=\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\dfrac{a_0}{a_n}$.

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Detailed Content	Time Ratio	Notes on Teaching
		The following properties should be studied in detail: (i) The number of distinct zeros of a non-zero polynomial is less than or equal to the degree of the polynomial. (ii) If the polynomial equation with integral coefficients has a rational root of the form $\dfrac{p}{q}$ where p and q are coprime integers, then p is an exact divisor of the constant term and q is an exact divisor of the leading coefficient of the polynomial. (iii) The condition for repeated (multiple) roots: For the polynomial equation $f(x) = 0$ to have $x = \alpha$ as a repeated root, it is necessary and sufficient that $f(\alpha) = 0$ and α is also a root of the equation $f'(x) = 0$ where $f'(x)$ denotes the derivative of $f(x)$. The more general form will be: For any positive integer k, α is a root of multiplicity k + 1 of the equation $f(x) = 0$ if and only if $f(\alpha) = 0$ and α is a root of multiplicity k of $f'(x) = 0$. and α is a root of multiplicity k + 1 of the equation $f(x) = 0$ if and only if α is a common root of $f(x) = 0$, $f'(x) = 0$ $\vdots$ $f^{(k)}(x)=0$ but not of $f^{(k+1)}(x)=0$ N.B. Complex roots occurring in conjugate pairs will be treated in the study of complex number in Unit A10.
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Unit A7: Vectors in R^2 and R^3

Objective: (1) To study the operations of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.

(2) To understand the concept of linearly dependent vectors and linearly independent vectors.

(3) To apply vectors in geometrical problems.

Detailed Content	Time Ratio	Notes on Teaching

7.1 Definition of Vectors and scalars — Time Ratio: 1

To begin with this unit, the difference between vectors and scalars should be explained to students. The representation of a vector, both pictorial and written, should be introduced. The current notations of vectors (such as $\overrightarrow{AB}$, **AB**, $\vec{a}$, **a**) and their magnitudes (such as $|\overrightarrow{AB}|$, |**AB**|, $|\vec{a}|$, |**a**|) are to be taught. The terms null vector, unit vector, equal vectors, negative vector, collinear vectors and coplanar vectors should also be defined.

7.2 Operations of vectors — Time Ratio: 7

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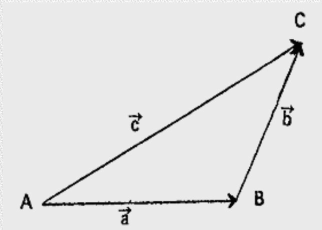
Student should know the laws of vector addition (namely, the triangle law, the parallelogram law, and the polygon law), the subtraction of vectors and the multiplication of a vector by a scalar.

(i) Triangle law

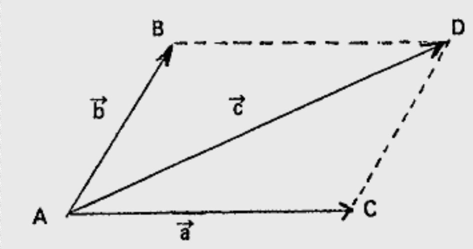


$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$
or
$$\vec{a}+\vec{b}=\vec{c}$$

It should be pointed out to the student that, when using the law to find $\vec{a}+\vec{b}$, the end point of vector $\vec{a}$ must coincide with the initial point of vector $\vec{b}$. It should be noted that the validity of the law still holds when A, B, C are collinear points.

Detailed Content	Time Ratio	Notes on Teaching

(ii) Parallelogram law



$$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$$
or
$$\vec{a}+\vec{b}=\vec{c}$$

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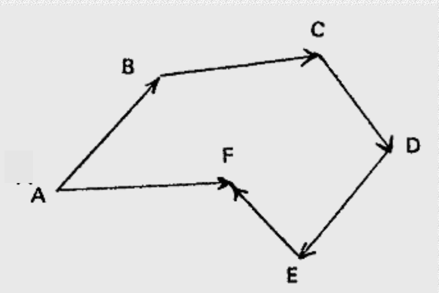
In a similar manner, teachers should remind the students that the initial points of vectors $\vec{a}$ and $\vec{b}$ must be coincident and in either of the above cases, $\vec{c}$ can also be regarded as the resultant of $\vec{a}$ and $\vec{b}$. The equivalence of the triangle law and the parallelogram law is worth discussing.

(iii) Polygon law



$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{AF}$$

The laws of the vector algebra like commutative law, associative law and distributive law should also be made known to students. The following diagrams may be useful in illustrating these properties.

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Detailed Content	Time Ratio	Notes on Teaching

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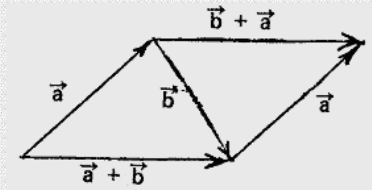
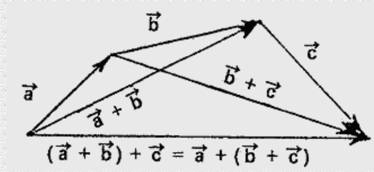
(a) Commutative law of addition: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$



(b) Associative law of addition: $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$



(c) Associative law for scalar multiplication:

$$(\alpha\beta)\vec{a}=\alpha\,(\beta\,\vec{a})$$

Distributive laws for scalar multiplication:

$$\alpha\,(\vec{a}+\vec{b})\;=\alpha\,\vec{a}+\alpha\,\vec{b}$$

$$(\alpha+\beta)\vec{a}=\alpha\,\vec{a}+\beta\,\vec{a}$$

After understanding the concept of scalar multiplication, students should have no difficulty to deduce the result that

if $\vec{a}$, $\vec{b}$ are non-zero vectors such that $\vec{a}=\alpha\,\vec{b}$ for some scalar α, then $\vec{a}\,//\,\vec{b}$.

It should be made clear to students concerning the resolution of a vector into component vectors, and the specification of a vectors as a sum of component vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. The resolution of vectors in $\mathbf{R}^2$ can be introduced with the following examples. In the first example, $\vec{r}$ is resolved into two components $5\vec{a}$ and $4\vec{b}$ in the directions of $\vec{a}$ and $\vec{b}$ respectively. This can be generalized to $\vec{r}=\alpha\,\vec{a}+\beta\,\vec{b}$ where $\vec{a}$ and $\vec{b}$ are non-collinear vectors in $\mathbf{R}^2$ and $\vec{r}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors in $\mathbf{R}^3$, for scalars α, β and γ.

Detailed Content	Time Ratio	Notes on Teaching

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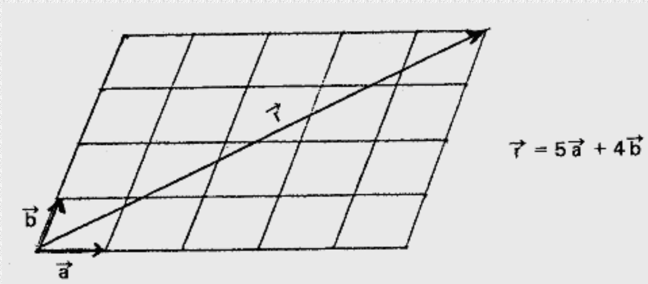
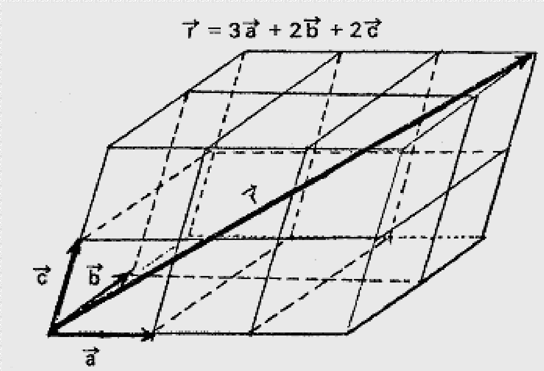
Examples:

1.



2.



Furthermore, scalar multiplication, addition and subtraction of vectors in terms of component vectors should be discussed.

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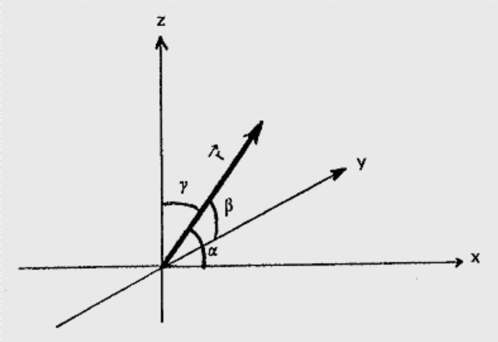
Detailed Content	Time Ratio	Notes on Teaching
7.3 Resolution of vectors in the rectangular coordinate system	2	The face that $\vec{i}$, $\vec{j}$ and $\vec{k}$ represent the unit vectors in the directions of the positive x-, y- and z-axis respectively and that any vector in $\mathbf{R}^2$ and $\mathbf{R}^3$ can be expressed in the form $a\vec{i}+b\vec{j}+c\vec{k}$ should be explained in detail. Students are required to be familiar with the following properties of vectors in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$: (i) $\left\|a\vec{i}+b\vec{j}+c\vec{k}\right\|=\sqrt{a^2+b^2+c^2}$; (ii) two vectors $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$ and $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$ are parallel if $a_1:b_1:c_1=a_2:b_2:c_2$. Moreover the meaning of direction ratio, direction cosines and direction angle of $\vec{r}$ should be explained with the help of diagrams, and the following properties should be discussed. (i) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ (ii) $\frac{\vec{r}}{\|\vec{r}\|}=\cos\alpha\,\vec{i}+\cos\beta\,\vec{j}+\cos\gamma\,\vec{k}$ 

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Detailed Content	Time Ratio	Notes on Teaching
7.4 Linear combination of vectors	4	The following definitions should be taught: (i) Let $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ be a set of vectors. An expression of the form $\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}$, where λ_1, λ_2, λ_3, …, λ_n are scalars, is called a linear combination of the vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$. If the scalars λ's are not all zero, it is called a non-trivial linear combination, otherwise it is a trivial linear combination. (ii) A set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is said to be linearly dependent if there exists a non-trivial linear combination of them equal to the zero vector. i.e. $\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}=\vec{0}$ where $\lambda_i\neq 0$ for some $i=1,2,3,\ldots,n$. (iii) A set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is said to be linearly independent if the only linear combination of them equal to zero is the trivial one. i.e. if $\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}=\vec{0}$ then $\lambda_1=\lambda_2=\lambda_3=\ldots=\lambda_n=0$. Students should be helped to deduce an immediate result from (ii) that the set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is linearly dependent if and only if one of the vectors is a linear combination of the others in the set. The geometrical significance of linear dependence of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ like the following should be elaborated. (i) vectors $\vec{r_1}$ and $\vec{r_2}$ of $\mathbf{R}^2$ are linearly dependent if and only if they are parallel; (ii) vectors $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ of $\mathbf{R}^3$ are linearly dependent if and only if they are coplanar.
7.5 Scalar (dot) product and vector (cross) product	6	The definition of the scalar product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\cdot\vec{b}$, in its usual context that $\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$, should be taught and the following properties discussed. 1. commutative law of scalar product: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ 2. distributive law of scalar product: $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ 3. $\vec{a}\cdot\vec{a}=\|\vec{a}\|^2$ 4. two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b}=0$ 5. $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$

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Detailed Content	Time Ratio	Notes on Teaching

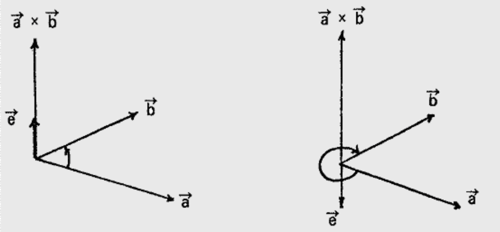
In particular, teachers should point out that, with references to the fifth property listed above, the scalar product can be used to find the angle between two vectors expressed in Cartesian components. In this connection students may be hinted to show that $\vec{i}\cdot\vec{i}=1$, $\vec{i}\cdot\vec{j}=0$ etc and the result that for $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$, $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$, $\vec{r_1}\cdot\vec{r_2}=a_1a_2+b_1b_2+c_1c_2$.

As for vectors product, the definition must be clearly provided. Special attention should be directed to the proper orientation of the right-hand system.

The vector product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\times\vec{b}$ is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\vec{e}$, where

(i) $\vec{e}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$;

(ii) θ is the angle from $\vec{a}$ to $\vec{b}$ measured in the direction determined by $\vec{e}$ according to the right-hand rule.



Discussion on the following properties is essential.

(i) $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$

(ii) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ and

$(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$

(iii) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

(iv) $|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$

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Students may be required to work out for themselves the results like $\vec{i}\times\vec{i}=\vec{0}$, $\vec{i}\times\vec{j}=\vec{k}$ etc as a prelude to deduce the result that for vectors expressed in Cartesian components $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$ and $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$

$\vec{r_1}\times\vec{r_2}=(b_1c_2-b_2c_1)\vec{i}+(c_1a_2-c_2a_1)\vec{j}+(a_1b_2-a_2b_1)\vec{k}$ or

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

Moreover, the properties that

(i) two non-zero vectors $\vec{a}$, $\vec{b}$ are parallel if and only if $\vec{a}\times\vec{b}=\vec{0}$

(ii) $|\vec{a}\times\vec{b}|$ may be interpreted as the area of the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$.

are helpful in reinforcing students' mastery of the concept.

7.6 Application of vectors in geometry	11	Teachers are advised to provide students with detailed explanation and adequate discussion as well as exemplification on the use of relative vectors including position vector and displacement vector.

The usual convention that the position vectors of points P and Q with respect to a reference point O are denoted by

$\overrightarrow{OP}$, $\overrightarrow{OQ}$ or $\vec{p}$, $\vec{q}$ respectively

and that

$\overrightarrow{PQ}=\vec{q}-\vec{p}$ should be highlighted.

The following results should be derived whilst other related generalization are also worth discussing for consolidation.

Position vector of point of division:

Let $\vec{a}$, $\vec{b}$ and $\vec{p}$ be the respective position vectors of A, B and P with reference to the point O. If P divides the line segment AB in the ratio of m:n then

$$\vec{p}=\frac{n\vec{a}+m\vec{b}}{m+n}$$

The different treatments for points of internal and external division should also be discussed. Sometimes the form $\vec{p}=\dfrac{k\vec{a}+\vec{b}}{1+k}$ should be preferred because, with adequate preparation on the part of the students this may be interpreted as the vector equation of the straight line passing through A and B. In particular, with respect to the Cartesian system with A being the point (x_1, y_1, z_1), B$(x_2, y_2, z_2)$and P(x, y, z), the two-point form of the line

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$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

can be easily obtained. At this juncture students may be asked to write down the direction number of the vector $\vec{b}-\vec{a}$ prior to the smooth generalization of the two-point form into

(i) symmetrical form

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ and}$$

(ii) parametric form

$x = x_1 + \ell$
$y = y_1 + m$
$z = z_1 + n$

where $\ell : m : n$ stands for the direction number of the line.

As a continuation, the equation of the plane having normal in the direction $\ell : m : n$ and passing through (x_1, y_1, z_1) can be introduced as an application of dot product:

$$\ell(x-x_1)+m(y-y_1)+n(z-z_1)=0$$

In this connection the general equation of a plane Ax + By + Cz + D = 0 should be introduced as a supplement with the following properties introduced.

(i) the direction ratios of the normal to the plane is A: B: C.

(ii) the perpendicular distance of the point P(x', y', z') to the plane is given by $\dfrac{Ax'+By'+Cz'+D}{\pm\sqrt{A^2+B^2+C^2}}$, where the sign is chosen so as to make the expression positive.

(iii) the angle θ between two planes

$\pi_1 : A_1x+B_1y+C_1z+D_1=0$ and

$\pi_2 : A_2x+B_2y+C_2z+D_2=0$ is

given by $\cos\theta=\dfrac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}$

(iv) $\pi_1 /\!/ \pi_2$ if and only if $\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$

$\pi_1 \perp \pi_2$ if and only if $A_1A_2+B_1B_2+C_1C_2=0$

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(v) the equation of the planes bisecting the angles between two planes π_1 and π_2 are

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2+B_1^2+C_1^2}}=\pm\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

Following the acquisition of the general knowledge of lines and planes, teachers may lead the students to appreciate the fact that

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases} \text{ represents}$$

the line of intersection of the planes π_1 and π_2 (if not parallel) and the direction ratios of the line can be found by

$$\begin{vmatrix}B_1 & C_1\\B_2 & C_2\end{vmatrix}:\begin{vmatrix}C_1 & A_1\\C_2 & A_2\end{vmatrix}:\begin{vmatrix}A_1 & B_1\\A_2 & B_2\end{vmatrix}$$

Furthermore the following properties between a line L with direction ratios p : q : r and a plane π: Ax + By + Cz + D = 0 should be discussed

(i) $L /\!/ \pi$ iff $Ap+Bq+Cr=0$

(ii) $L \perp \pi$ iff $\dfrac{A}{p}=\dfrac{B}{q}=\dfrac{C}{r}$

(iii) the angle θ made between L and π is given by

$$\sin\theta=\left|\frac{Ap+Bq+Cr}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{p^2+q^2+r^2}}\right|$$

The conditions for two lines to be coplanar should be also studied i.e. two lines are coplanar if and only if they intersect or are parallel.

Suppose L_1 is $\dfrac{x-a_1}{p_1}=\dfrac{y-b_1}{q_1}=\dfrac{z-c_1}{r_1}$

L_2 is $\dfrac{x-a_2}{p_2}=\dfrac{y-b_2}{q_2}=\dfrac{z-c_2}{r_2}$,

L_1 and L_2 are coplanar iff $\begin{vmatrix}a_1-a_2 & p_1 & p_2\\b_1-b_2 & q_1 & q_2\\c_1-c_2 & r_1 & r_2\end{vmatrix}=0$

Throughout this sub-unit, teachers are encouraged to apply vector approach as far as possible in deducing the above-mentioned properties or results. In particular the use of dot product to find the projection of a vector $\vec{p}$ along a vector $\vec{r}$ and the use of cross product to evaluate the area of triangle with vertices given should be explained.

Time Ratio: 31

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Unit A8: Matrices

Objective: (1) To learn the concept and operations of matrices.
(2) To learn the properties and operations of square matrices of order 2 and 3 and their determinants.
(3) To apply matrices to two dimensional geometry.

Detailed Content	Time Ratio	Notes on Teaching
8.1 Matrices and their operations	4	The general form of a matrix with m rows and n columns, namely an m $\times$ n matrix, should be introduced. Students should know the operations: addition, subtraction, multiplication and scalar multiplication of matrices and study their properties. The fact that, in general, AB $\neq$ BA holds for matrices A and B should be mentioned and explained. Terms like zero matrix, identity matrix and the transpose of a matrix should be introduced.
8.2 Square matrices of order 2 and 3	9	The definition of square matrices and their determinants should be defined. The concepts and uses of singular and non-singular matrices should also be made clear to students. Students should be able to evaluate determinants of square matrices and find the inverses for non-singular matrices. They are also expected to have knowledge of simple properties of inverses and determinants like: A. Properties of inverse (i) The inverse of a matrix is unique. (ii) A square matrix has inverse if and only if it is non-singular. (iii) If A-is non-singular, then AB = 0 implies B = 0. (iv) If A is non-singular, then AB = AC implies B = C. (v) If A, B are non-singular, λ is a non-zero scalar and n is a positive integer, then AB, A^{-1}, A^t, λA, A^n are non-singular and $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$, $(A^t)^{-1} = (A^{-1})^t$, $(\lambda A)^{-1} = \lambda^{-1}A^{-1}$, $(A^n)^{-1} = (A^{-1})^n$. B. Properties of determinant (i) If two rows (or columns) of a determinant are identical or proportional, the value of the determinant is zero. (ii) The interchange of two rows (or columns) changes the sign of the determinant without altering its numerical value.

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Detailed Content	Time Ratio	Notes on Teaching
		(iii) The value of a determinant is unaltered by changing its rows into columns (or vise versa), i.e. det A^t = det A or $\lvert A^t\rvert = \lvert A\rvert$ (iv) If every element in any row (column) is multiplied by the same number, then the value of the determinant is multiplied by that number. (v) The determinant of the product of two square matrices of the same order is equal to the product of the determinants of the matrices, i.e. det AB = det A · det B or $\lvert AB\rvert = \lvert A\rvert\lvert B\rvert$
8.3 Applications to two dimensional geometry	8	Students should be familiar with matrix representation of a point and a vector and, furthermore, reflections, rotations, enlargements, shears, translations and their compositions. A few examples of such transformations, represented by 2 x 2 matrices are given below: (i) Reflection in the line $y = (\tan\theta)x$ is given by the matrix A= $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ (ii) Rotation through an angle ϕ about the origin is given by the matrix B $= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ (iii) Enlargement about origin with scale factor k$\neq$ 0 is given by the matrix C $= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ (iv) Shear parallel to x-axis with factor k is given by the matrix D = $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ For (iii) and (iv), the effect on shape.and area should be discussed. It should be made clear to students that under all these transformations of the plane, each point P(x, y) will be transformed to a new point P'(x', y') satisfying $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ where T stands for a transformation. Elaboration on the composition of the transformations is essential to enable students to have a thorough understanding on matrix multiplication. *Example:* The effect of reflection in the line y = (tanθ) x followed by a rotation through an angle θ about the origin is given by the product BA complying with the convention BA $\binom{x}{y} = \binom{x'}{y'}$ as mentioned above. It should be emphasized that the product is to be interpreted from right to left: first apply transformation A, then apply transformation B.

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Detailed Content	Time Ratio	Notes on Teaching
		Other composition of transformations like the following may be mentioned: *Example:* The transformation having equations $\begin{cases} x' = y + 2 \\ y' = -x + 4 \end{cases}$ whose representation in matrix form is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ The transformation with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a clockwise rotation about the origin through 90°. Hence, it could be viewed as a rotation followed by a translation, however, with (3, 1) as the invariant point the transformation can be regarded as a clockwise rotation about (3, 1) through 90°.
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Unit A9: System of Linear Equations in 2 or 3 Unknowns

Objective: (1) To solve a system of linear equations using Gaussian elimination.
(2) To recognize the existence and uniqueness of solution.

Detailed Content	Time Ratio	Notes on Teaching
9.1 Gaussian elimination and Echelon form	5	A matrix which satisfies the following 2 properties is said to be in Echelon form: (1) The 1st k rows are non-zero; the other rows are zero. (2) The 1st non-zero element in each non-zero row is 1, and it appears in a column to the right of the 1 st non-zero element of any preceding row. *Example:* The following 5 x 8 matrix is in Echelon form: $\begin{pmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ Students should be able to solve a system of linear equations in two or three unknowns by using Gaussian elimination, which reduces a matrix in Echelon form by elementary operations on its rows. *Example:* Solve the system $\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + x_2 - 2x_3 = 1 \\ 4x_1 - 3x_2 - x_3 = 3 \\ 2x_1 + 4x_2 + 2x_3 = 4 \end{cases}$ The augmented matrix $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ The original system of equations is equivalent to the system of equations $\begin{cases} x_1 \quad - x_3 = 0 \\ \quad x_2 + x_3 = 1 \\ \quad\quad x_3 = 1 \end{cases}$ which gives $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.

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Detailed Content	Time Ratio	Notes on Teaching
9.2 Existence and uniqueness of solution	5	Students should be able to know the conditions for the existence and uniqueness of solution for a system of linear equations in two or three unknowns. For a system of linear equations in two unknowns: $a_1x + b_1y = d_1$ $a_2x + b_2y = d_2$ (i) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$, the system has unique solution. Geometrically, the equations represent a pair of intersecting straight lines. (ii) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ and $\det\begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} \neq 0$, the system has no solution. Geometrically, the equations represent a pair of parallel (but not coincident) straight lines. (iii) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ and $\det\begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} = 0$, the system has infinite number of solutions. Geometrically, the equations represent a pair of coincident straight lines. For systems of equation in three unknowns, examples like the following should be mentioned. The corresponding geometrical meaning may be discussed if the students have grasped some ideas of three dimensional coordinate geometry. (i) In solving the equations $\begin{cases} 2x + y - z = 7 \\ 5x - 4y + 7z = 1 \\ 7x - 3y + 6z = 8 \end{cases}$, it is obvious that the third one is redundant. Teachers may discuss with the students on the method to obtain the solution $x = \dfrac{29 - 3\lambda}{13}, y = \dfrac{33 + 19\lambda}{13}, z = \lambda$ where λ is arbitrary. (ii) Solving equations like $\begin{cases} x + y + z = 3 \\ 2x - 3y + 2z = 1 \\ 3x - 2y + 3z = 7 \end{cases}$ which are inconsistent. Following this manner, the conditions for the existence and uniqueness of solution for a system of equations in three unknowns may be given in more abstract terms.
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Unit A10: Complex Numbers

Objective: (1) To learn the properties of complex numbers, their geometrical representations and applications.
(2) To learn the De Moivre's Theorem and its applications in finding the nth roots of complex numbers, in solving polynomial equations and proving trigonometric identities.

Detailed Content	Time Ratio	Notes on Teaching
10.1 Definition of complex numbers and their arithmetic operations	3	A short introduction of the symbol i should be given. The number z = x + yi, where x, y are real, is called a complex number and x and y are known respectively as its real (Re z) and imaginary (Im z) parts. When x = 0, y $\neq$ 0, z = yi is said to be purely imaginary and when y = 0, z = x is real. Students may be asked what definition should be adopted for the equality of complex numbers, however there is no ordering property for complex numbers. The sum, difference, product and quotient of two complex numbers should be defined.
10.2 Argand diagram, argument and conjugate	6	Students are expected to know the definitions of the terms modulus |z|, argument arg z, principal (value of) argument (or amplitude) and conjugate $\bar{z}$ of a complex number z. The complex number z = r(cosθ + isinθ), in the modulus — argument form (polar form), can be written as z = rcisθ. Students are expected to know the following properties of complex numbers: (i) $\|z_1 z_2\| = \|z_1\|\|z_2\|$ (ii) arg $z_1 z_2$ = arg z_1 + argz_2 +2kπ where k is an integer (iii) $\left\|\dfrac{z_1}{z_2}\right\| = \dfrac{\|z_1\|}{\|z_2\|}$ (iv) $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2 + 2k\pi$ where k is an integer and $z_2 \neq 0$. Properties about conjugate complex numbers should be taught: 1. $\bar{\bar{z}} = z$ 2. $\bar{z}$ = 0 iff z=0 3. A complex number is self-conjugate (conjugate to itself) iff it is real. 4. $z \cdot \bar{z} = \|z\|^2$

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Detailed Content	Time Ratio	Notes on Teaching
		5. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ 6. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ 7. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ 8. $\overline{(\frac{z_1}{z_2})} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$ From these properties students can easily prove that if α is a root of a polynomial equation with real coefficients, then $\overline{\alpha}$ is also a root. i.e. in a polynomial equation with real coefficients, roots which are not real occur in conjugate pairs. Students are expected to know the following inequalities: (i) $\lvert \text{Re } z\rvert \leq \lvert z\rvert$ (ii) $\lvert \text{Im } z\rvert \leq \lvert z\rvert$ (iii) $\lvert z_1 + z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$ (Triangle inequality) Students can also be asked to prove in a similar way that $\lvert z_1 - z_2\rvert \geq \lvert z_1\rvert - \lvert z_2\rvert$ and $\lvert z_1 - z_2\rvert \geq \lvert z_2\rvert - \lvert z_1\rvert$. The triangle inequality can easily be extended by induction to $\lvert z_1 + z_2 + \ldots + z_n\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert + \ldots + \lvert z_n\rvert$. The geometrical representation of complex numbers in an Argand diagram should be studied. Students should know the terms real axis and imaginary axis. The representation of a complex number in polar form and its geometrical meaning should also be taught. The notation $e^{i\theta}$ for $\text{cis}\theta$ may be introduced so that $z = re^{i\theta}$. The notation is known as the exponential form or the Euler form of a complex number.
10.3 Simple applications in plane geometry	5	Students are expected to know the geometrical meaning of the triangle inequality. Various uses of complex number in plane geometry should be studied. The following are two examples: 1. In the Argand diagram, XYZ is an equilateral triangle whose circumcentre is at the origin. If X represents the complex number 1 + i, find the numbers represented by Y and Z. 2. If z_1, z_2 and z_3 are three distinct complex numbers denoting the vertices of an equilateral triangle, then $z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$.

Detailed Content	Time Ratio	Notes on Teaching
		Examples on the loci of points moving on the Argand plane should be studied. Two simple examples are given below: (i) To find the locus of the point z such that $\lvert z - a\rvert = k$, where a is a complex number and k is a positive constant. (ii) To find the locus of the point z which moves such that $\left\lvert \frac{z-a}{z-b} \right\rvert = k$, where a, b are complex numbers, for various values of the positive constant k.
10.4 De Moivre's theorem		
10.4a De Moivre's theorem for rational indices	3	Students should learn how to prove the theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ when n is a positive integer with the assumption that $(\cos\theta + i\sin\theta)^0 = 1$. When n is a negative integer, by putting $n = -m$ where m is a positive integer, students should be able to prove that the theorem is also true for negative integer n. However for the case $n = \frac{p}{q}$, where p, q are integers and $q \neq 0$, the proof may be provided afterwards till the students have acquired the knowledge of the n^{th} roots of a complex number.
10.4b Applications to trigonometric identities	3	By De Moivre's theorem, $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ where n is positive integer, expressions for $\cos n\theta$ and $\sin n\theta$ can be obtained in terms of the powers of $\cos\theta$ and $\sin\theta$. By considering $z = \cos\theta + i\sin\theta$, then expressions $\begin{cases} z + \frac{1}{z} = 2\cos\theta \\ z - \frac{1}{z} = 2i\sin\theta \end{cases}$ and $\begin{cases} z^n + \frac{1}{z^n} = 2\cos n\theta \\ z^n - \frac{1}{z^n} = 2i\sin n\theta \end{cases}$

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Detailed Content	Time Ratio	Notes on Teaching
		can be used to express powers of $\cos\theta$ and $\sin\theta$ in terms of sines and cosines of multiples of θ. For example, students should be able to express $\cos^4\theta\sin^3\theta$ as a sum of sines of multiples of θ and $\cos^3\theta\sin^4\theta$ as a sum of cosines of multiples of θ.
10.4c n^{th} roots of a complex number and their geometrical interpretation	~~5~~ 4	Students should learn the meaning of the n^{th} roots of a complex number. The n^{th} roots of unity should be studied in detail. Several examples can be discussed in class: 1. To find the fifth roots of –1. 2. To solve the equation $z^4+z^3+z^2+z+1=0$. 3. To find the cube roots of 1 + i. 4. Factorize $z^{2n}-2z^n\cos n\theta+1$ into real quadratic factors.
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Unit B1: Sequence, Series and their Limits

Objective: (1) To learn the concept of sequence and series.
(2) To understand the intuitive concept of the limit of sequence and series.
(3) To understand the behaviour of infinite sequence and series.

Detailed Content	Time Ratio	Notes on Teaching
1.1 Sequence and series	6	Clear concepts of sequence and series should be provided. The following suggested versions may be adopted: If a_n is a function of n which is defined for all positive integral values of n, its values a_1, a_2, a_3, …, a_n, … are said to form a sequence. The sequence is finite or infinite according to the numbers of terms of it being finite or infinite. Furthermore $a_1 + a_2 + \ldots + a_n + \ldots$ is said to form a series. Likewise, it is finite or infinite according to the numbers of terms contained. The notation $S_n=\sum_{r=1}^{n}a_r$ or $\sum_{1}^{n}a_r$ is commonly used. Some simple rules concerning the operations of sequences and series may be introduced. For the sake of convenience, denote the sequences a_1, a_2, a_3, … and b_1, b_2, b_3, … by $\{a_i\}$ and $\{b_i\}$, then (i) $\{a_i\}\pm\{b_i\}=\{a_i\pm b_i\}$ (ii) $\lambda\{a_i\}=\{\lambda a_i\}$, viz, the idea of termwise operations may be touched upon. Regarding series, the following methods of summation should be discussed. (1) Mathematical induction: already dealth with in Unit A3. (2) Method of difference: teachers should amplify in the expressing the rth term of the series as the difference of f(r + 1) and f(r) where f(x) is a function of x. i.e. if $a_r = f(r+1) - f(r)$ then $\sum_{1}^{n}a_r=\sum_{1}^{n}(f(r+1)-f(r))$ $= f(n+1) - f(1)$. Some typical examples are $\sum_{1}^{n}\frac{1}{r(r+1)}$ and $\sum_{1}^{n}r(r+1)$.

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Detailed Content	Time Ratio	Notes on Teaching
		For series whose terms are presented in a recurrence of the form $a_r - a_{r-1} = f(r)$ or $a_r = Aa_{r-1} + Ba_{r-2}$ some basic methods should be introduced, especially for the latter one the following approach may be discussed: Suppose α, β are the roots of the auxiliary equation $\lambda^2 = A\lambda + B$, (i) if $\alpha \neq \beta$, $a_r = k_1\alpha^r + k_2\beta^r$; (ii) if $\alpha = \beta$, $a_r = (k_1 + rk_2)\alpha^r$ where k_1 and k_2 are constants to be determined.
1.2 Limit of a sequence and series	7	The concept of the limit of a sequence should be taught with an intuitive approach. The following version may be considered: Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence. If for all sufficiently large values of n, the difference between a_n and a constant ℓ is as small as we please, we say that $a_n \to \ell$ when $n \to \infty$ or $\lim_{n\to\infty} a_n = \ell$. Teachers should emphasize on the following points: (i) ℓ is called the limit of the sequence; (ii) the limit ℓ, if exists, is unique; (iii) the sequence is said to converge to ℓ or the sequence is convergent with limit ℓ; (iv) if a sequence does not converge to any limit, it is said to be divergent. Ample examples illustrating convergence and divergence should be provided. (1) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for a>0. (2) The sequence $a_n = \dfrac{\sin \frac{1}{2} n\pi}{n}$ which converges to the limit 0. (3) The divergent sequence $a_n = \{1 + (-1)^n\}\sqrt{n}$. (4) The divergent oscillatory sequence $a_n = (-1)^n(1 + \frac{1}{n})$. N.B. Sequences could be classified as convergent, divergent (to $+\infty$ or $-\infty$) or oscillatory (does not converge nor diverge (to $+\infty$ or $-\infty$)). Some common properties of convergent sequence should be included in the discussion with students:

Detailed Content	Time Ratio	Notes on Teaching
		Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ be convergent sequences with limits be a and b respectively, the following sequence are also convergent: (i) $\lambda a_1, \lambda a_2, \lambda a_3, \ldots$ converges λa, where λ is a constant. (ii) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ converges to a + b. (iii) $a_1b_1, a_2b_2, a_3b_3, \ldots$ converges to ab. (iv) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ converges to $\frac{a}{b}$ provided $b \neq 0$. Finally, students should be led to appreciate the following results that (i) for the convergent sequence $a_1, a_2, a_3, \ldots$ with limit a, $\lim_{n\to\infty} a_{n+k} = \lim_{n\to\infty} a_n = a$, where k is a positive integer. (ii) for the two convergent sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ with the same limit ℓ and if a sequence $c_1, c_2, c_3, \ldots$ such that $a_i \leq c_i, \leq b_i$ when i > k for some positive integer k, then $c_1, c_2, c_3, \ldots$ also converges and to the same limit ℓ. This property is commonly known as the Sandwich Theorem. Teachers may also touch upon the meaning of monotonic sequence and bounded sequence to broaden students' understanding. As for infinite series, a parallel treatment could be provided as follows: (1) Concept of convergence The series $u_1 + u_2 + u_3 + \ldots$ is convergent if $\lim_{n\to\infty}\sum_1^n u_i = S$ exists and the series is said to be convergent to the limit. (Sometimes S may be called the sum of the series.) If S_n represents $u_1 + u_2 + \ldots + u_n$, then the result may be stated as $S_n \to S$ as $n \to \infty$ or $\lim_{n\to\infty} S_n = S$. ($S_n = u_1 + u_2 + \ldots + u_n$ is commonly known as the n^{th} partial sum). And, in a more or less the same situation, divergent series and/ or oscillatory series may be introduced subject to teachers' preference. (2) Properties of convergent series $u_1 + u_2 + u_3 + \ldots$ with limit S and $v_1 + v_2 + v_3 + \ldots$ with limit S' then (a) $\lambda u_1 + \lambda u_2 + \lambda u_3 + \ldots$ converges to λS where λ is a constant. (b) $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3)$ converges to S + S'. (c) If $u_1 + u_2 + u_3 + \ldots$ Is convergent, then $\lim_{n\to\infty} u_n = 0$

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Detailed Content	Time Ratio	Notes on Teaching
1.3 Convergence of a sequence and series	5	Further properties of convergent sequence like (i) convergent sequences are bounded (ii) a monotonic and bounded sequence is convergent should be introduced. Some typical convergent and divergent sequences should be discussed so as to illustrate the method in finding limits of sequences. The following examples may be considered: (A) Convergent sequences (i) $a_n = x^n$ with $\lvert x\rvert < 1$ (ii) $a_n = \sqrt[n]{n}$ (iii) $a_n = \frac{x^n}{n!}$ (B) Convergent series (i) $r + r^2 + r^3 \ldots$ with $\lvert r\rvert < 1$ (ii) $1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots$ (iii) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$ (iv) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ (C) Divergent series (i) $\sum\frac{1}{n}$ (ii) $\sum(1-\frac{1}{n})^n$ (iii) $\sum\frac{1}{\sqrt{n}}$ Some typical applications of the Sandwich Theorem should be included for illustration whereas convergence tests of series are not required.
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Unit B2: Limit, Continuity and Differentiability

Objective: (1) To understand the intuitive concept of the limit of a function.
(2) To understand the intuitive concept of continuity and differentiability of a function.
(3) To recognize limit as a fundamental concept in calculus.

Detailed Content	Time Ratio	Notes on Teaching
2.1 Limit of a function	5	An intuitive understanding of the concept of limit of function is expected. As a matter of fact, the concept of the limit of a function y = f (x) at the point x = a can be related to the concept of the limit of a sequence. This is done by allowing the independent variable to run through a convergent sequence of numbers $\{x_n\}$ tending to the limit a (the abscissa sequence), and considering the ordinate sequence $\{f(x_n)\}$. Thus a more vivid visualization of the fact that $\{f(x_n)\}$ tends to a finite value ℓ as $\{x_n\}$ tends to a could be established i.e. $f(x) \to \ell$ when $x \to a$ or $\lim_{x\to a} f(x) = \ell$. Some teachers may perhaps prefer just to focus students' attention to the fact that the difference between f(x) and ℓ can be made arbitrarily small when x is sufficiently close to a so as to reinforce the idea that $f(x) \to \ell$ when $x \to a$. It must be pointed to students that, from the existence of the value f(a) of the function, one can certainly not conclude that the limit $\lim_{x\to a} f(x)$ must also exist and be equal to f(a), though this is very often the case. The following example may be considered: $f(x) = \begin{cases} 1 & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$ in which f(0) = 0 and $\lim_{x\to 0} f(x) = 1$ It may be important in the passage to the limit whether the independent variable approaches the value a in the sense of increasing values of x, that is, from the left, or in the sense of decreasing values of x, that is from the right. In these cases, the limits are referred to, respectively, as the left-hand limit, usually denoted by $\lim_{x\to a^-} f(x)$, and the right-hand limit $\lim_{x\to a^+} f(x)$. In this context, students could be led easily to appreciate that the function f(x) has a limit as $x \to a$ if and only if the left-hand and right-hand limits as $x \to a$ are equal. For a more comprehensive understanding of limit, teachers should touch upon the case when $x \to \infty$ by reiterating that the difference between f(x) and ℓ could be made arbitrarily small when x is sufficiently large. Symbolically, it is presented as $\lim_{x\to\infty} f(x) = \ell$.

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Detailed Content	Time Ratio	Notes on Teaching
		The following properties of the limit of a function should be included in discussion: (i) $\lim\limits_{x\to a}[f(x)+g(x)]=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)$ (ii) $\lim\limits_{x\to a}[kf(x)]=k\lim\limits_{x\to a}f(x)$, k is independent of x (iii) $\lim\limits_{x\to a}[f(x)\cdot g(x)]=\lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x)$ (iv) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$ provided $\lim\limits_{x\to a}g(x)\neq 0$ (v) If $f(x)\le h(x)\le g(x)$ holds when x is close to a and $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=\ell$ then $\lim\limits_{x\to a}h(x)=\ell$ also. Some important limits, like the following, should be introduced: (1) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ (2) $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=e$ (3) $\lim\limits_{x\to\infty}(1+\frac{a}{x})^x=e^a$ (4) $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$ (5) $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ell n\, a$; a > 0 (6) $\lim\limits_{x\to 0}\frac{\ell n(1+x)}{x}=1$ (7) $\lim\limits_{x\to 1}\frac{\ell n\, x}{x-1}=1$ Adequate practice in evaluating the limit of function should be provided for consolidation.

Detailed Content	Time Ratio	Notes on Teaching
2.2 Continuity of a function	4	Continuity should be defined on the basis of the limit of function with an intuitive approach; the $\varepsilon-\delta$ approach may not be desirable. The following suggested version may be considered: A function f(x) is continuous at x = a if $\lim\limits_{x\to a}f(x)$ exists and is equal to f(a). A function is continuous in an interval if it is continuous at every point of the interval. Some common functions like (i) $f(x)=x^2$ which is continuous in every interval; (ii) $f(x)=\frac{1}{x-1}$ which is not continuous in the whole interval $0\le x\le 5$ should be discussed as a prelude to introduce the concept of point of discontinuity. It should be noted that just informal treatment on this concept is expected, however, teachers are advised to provide students with a good spectrum of examples as a form of reinforcement. Furthermore, the fact that the sum, difference and product of two functions continuous at x = a are likewise continuous at this point. Their quotient is continuous provided that the denominator is not zero at x = a. Teachers may quote a lot of everywhere continuous functions to initiate students' further study on this topic: (i) polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$ (ii) exponential function $f(x)=a^x$; a > 0 (iii) logarithmic function $f(x)=\log_a x$; a > 0, $a\neq 1$ (iv) trigonometric functions like sinx, cosx. Concerning the continuity of composite function, teachers may consider the suggested version : Let y = f[g(x)] be a composite function, when inner function g(x) is continuous at x = a and whose outer function y = f(t) is continuous at t = g(a), then the composite function y = f[g(x)] is continuous at x = a. Also teachers may highlight the fact that every continuous function of a continuous function is again continuous. Teachers should point out that functions which are continuous in an interval form a class of functions with noteworthy properties, like the following, and the discussion of them is expected but formal proof of them is not desirable.

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Detailed Content	Time Ratio	Notes on Teaching
		(i) If a function f(x) is continuous in a closed interval [a, b] with f(a) = A and f(b) = B where $A \neq B$, then f(x) takes every value between A and B at least once. (The Intermediate Value Theorem). (ii) A function that is continuous in a closed interval is bounded there. (iii) A function that is continuous in a closed interval attains maximum and minimum values in the interval. (Properties (ii) and (iii) are known as Weierstrass Theorem).
2.3 Differentiability of a function	4	Regarding the differentiability of a function f(x) at the point x = a the following version may be considered. A function f(x) is said to be differentiable at the point x = a if and only if the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ or $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists Teachers may also at the same time put forth the idea that if a function is differentiable at a certain point, it is also continuous there and that continuity is a necessary condition for differentiability but not a sufficient one. Moreover, the definition of the derivative of a function at x = a, being the value of the above limit, can be taught very smoothly following students' acceptance of the idea of differentiability. The common notations for the derivative of f(x) at x = a like $f'(a), \frac{d}{dx} f(x) \Big\vert_{x=a}$ and $\frac{dy}{dx} \Big\vert_{x=a}$ should be mentioned. Teachers may also touch upon the differentiability of a function in the whole interval in the context that the .derivative of the function exists for all points in that interval. Furthermore, teachers should elaborate on the property that to each value x in the interval, there corresponds the derivative f'(x) of the function at the point x; thus f'(x) is again a function of x and is called the derived function of f(x). At this stage, ample examples should be worked out to reinforce students' mastery of the concept and skills concerning differentiation. In particular, examples to find the derivative of different typical functions from the first principles are of particular importance. In this connection, adequate practices are indispensable. The following examples are typical:

Detailed Content	Time Ratio	Notes on Teaching
		(1) Find the derivative of the functions from the first principles (i) x^2 at x = 1 (ii) e^x at x = 0 (iii) sinx at $x = \pi/4$ (2) Differentiate, from the first principles, the following functions (i) $f(x) = x^n$ where n is a positive integer (ii) $f(x) = e^x$
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Unit B3: Differentiation

Objective: (1) To acquire different techniques of differentiation.
(2) To learn and acquire techniques to find higher order derivative.
(3) To understand the intuitive concept of Rolle's Theorem and Mean Value Theorem.

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Detailed Content	Time Ratio	Notes on Teaching
3.1 Fundamental rules for differentiation	4	As a continuition, the following rules should be taught: (1) $\frac{d}{dx}(k)=0$, where k is a constant (2) $\frac{d}{dx}(x^r)=rx^{r-1}$, where r is real (3) $\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$ (4) $\frac{d}{dx}[kf(x)]=k\frac{d}{dx}f(x)$, where k is a constant (5) $\frac{d}{dx}[f(x)g(x)]=g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x)$ (product rule) (6) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{g(x)^2}$ $g(x)\neq 0$ (quotient rule) Proofs of the above rules should be mentioned or presented as a form of practice in order to strengthen students' mastery of the concept and skill. From (3) to (6), the existence of the derivatives of f(x) and g(x) should be emphasized. Regarding (2), a proof for r being integral will be enough while for the general case r being real the proof may be provided at a later stage till the students have learnt Chain rule. Typical examples in using the above rules to obtain derivative of various common functions should be done for illustration.
3.2 Differentiation of trigonometric functions	2	Differentiation of the following functions should be taught: 1. sinx 2. cosx

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Detailed Content	Time Ratio	Notes on Teaching
		3. tanx 4. cosecx 5. secx 6. cotx Students may be encouraged to do the proof themselves under teachers' supervision and, in particular, they should be reminded to derive the results for (4) to (6) using the quotient rule.
3.3 Differentiation of composite functions and inverse functions	4	For a composite function y = f[g(x)], the derivative is obtained through the chain rule $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$ or = f'(t)g'(x) with t = g(x) For the inverse function $x = f^{-1}(y)$ of y = f(x), the derivative is obtained through $\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$ It is suggested that examples like $\frac{d}{dx}(\sin^{-1}x)$, $\frac{d}{dx}(\cos^{-1}x)$, $\frac{d}{dx}(\tan^{-1}x)$ and $\frac{d}{dx}(x^{-n})$, $\frac{d}{dx}(x^{\frac{1}{n}})$ with n being positive integer may be used for illustration.
3.4 Differentiation of implicit functions	2	It is often necessary to differentiate a function defined implicitly by F(x, y) = 0. This is done by differentiating both sides of the given equation with respect to the independent variable x and applying the rules mentioned above. Various illustrating examples should be included to enrich the discussion. The following are some suggestions. (i) If $x\cos y^3+y\sin 2x=1$, find $\frac{dy}{dx}$ (ii) Given $2x^2-y^2+12x-2y+3=0$, find $\frac{dy}{dx}$ at the point (2, 5). (iii) Find $\frac{dy}{dx}$ for $\cos(x^2-y^2)=xy$.

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Detailed Content	Time Ratio	Notes on Teaching
3.5 Differentiation of parametric equations	2	A parametric representation of a function y = f(x) is given by x = u(t) and y = v(t). Hence y can be expressed as a composite function of the parameter t in the form y = f(u(t)). By applying the chain rule for differentiation, the result $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ and hence $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ or $f'(x)=\frac{v'(t)}{u'(t)}$ can be obtained. It should be clarified that in this derivation it is assumed that u(t) and v(t) are differentiable and $u'(t) \neq 0$. Typical examples for illustration include finding $\frac{dy}{dx}$ for the following functions: (i) the ellipse x = acost, y = bsint (ii) the cycloid. x = a(t-sint), y = a(1 -cost)
3.6 Differentiation of logarithmic and exponential functions	6	The following rules should be taught and their proofs may be provided with the suggested approach. 1. $\frac{d}{dx}(\ell n\, x)=\frac{1}{x}$ (using $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$) 2. $\frac{d}{dx}e^x=e^x$ (using $\frac{d}{dx}(\ell n\, y)=\frac{1}{y}$ and chain rule, where $y=e^x$, or applying the rule about the derivative of inverse function) 3. $\frac{d}{dx}(\log_a x)=\frac{1}{x\,\ell n\, a}$ 4. $\frac{d}{dx}(a^x)=a^x\ell n\, a$ Examples provided should include functions of the types like e^{x^3} and $\log_a\sqrt{x^2+1}$. (N.B. At this juncture the proof for the formula $\frac{d}{dx}x^n=nx^{n-1}$ when n is rational and when n is real may be mentioned for the sake of completeness.)

Detailed Content	Time Ratio	Notes on Teaching
		Teachers should also highlight some common applications of logarithmic differentiation as follows: when y is a complicated function of x and especially when it involves a variable as index, the value of $\frac{dy}{dx}$ may sometimes be more easily obtained by logarithmic differentiation. Typical examples of this kind include functions like $y = x^x$ and $y=\frac{(x+a)(x+b)}{(x+c)(x+d)}$.
3.7 Higher order derivatives and Leibniz's Theorem	5	The definition of higher order derivatives and the symbols f"(x), $f^{(n)}(x)$, $\frac{d^ny}{dx^n}$ should be introduced. Also the abilities to find higher order derivatives of functions given in parametric form and to apply the Leibniz's Theorem, viz $\frac{d^n}{dx^n}(uv)=\sum_{r=0}^{n}C_r^n u^{(r)}v^{(n-r)}$ are expected. Students may attempt to prove the theorem by mathematical induction. Examples showing the use of Leibniz's Theorem in obtaining relations involving higher order derivatives especially of implicit functions should be illustrated. Examples of this kind include 1. Find the n^{th} derivatives of $\cos^2 x\sin x$ and $x^3\cos x$. 2. Let $f(x) = \tan^{-1}x$, show that $(1 + x^2) f''(x) + 2x f'(x) = 0$ and hence obtain the n^{th} derivative of f(x) for x = 0. 3. Given $f(x)=\frac{x^3}{x^2-1}$ show that $f^{(n)}(0)=\begin{cases}0 \text{ if n is even}\\ -n! \text{ if n is odd}\end{cases}$ where n is an integer and $n \geq 3$
3.8 The Rolle's Theorem and Mean Value Theorem	3	The intuitive concept of the Rolle's Theorem and Mean Value Theorem as well as their geometrical interpretation should be taught. For abler students the proof may be mentioned. Simple and straightforward applications of the theorems are expected. The following examples may be considered:

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Detailed Content	Time Ratio	Notes on Teaching
		1. If $f'(x) = 0$ for all x in an interval, then f(x) is constant in that interval. 2. If $f'(x) = g'(x)$ for all x in an interval, then f(x) and g(x) differ in that interval by a constant. 3. Prove that if $\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$ then the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root between 0 and 1
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Unit B4: Application of Differentiation

Objective: (1) To learn and to use the L' Hospital's Rule.
(2) To learn the applications of differentiation.

Detailed Content	Time Ratio	Notes on Teaching
4.1 The L' Hospital's Rule	4	Limits having the following indeterminate forms should be introduced: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞ Accompanying examples illustrating the type mentioned are highly desirable. The L' Hospital's Rule $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ for the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ should be taught in the first place. The examples that follow may be considered: (1) $\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$ (2) $\lim_{x \to a^+} \frac{\ell n \sin(x-a)}{\ell n \tan(x-a)}$ Teachers should emphasize that $\frac{f'(x)}{g'(x)}$ should be simplified before taking limit and the process can be repeated until $\lim_{x \to a} \frac{f^{(m)}(x)}{g^{(m)}(x)}$ is obtained in a non-indeterminate form. As for the other indeterminate forms, examples should be worked out showing that they can be expressed in the determinate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that the rule may be applied. The following examples may be considered: (1) $\lim_{x \to 0} (\frac{1}{x} - \cot x)$ (2) $\lim_{x \to 0^+} x^x$ (3) $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ The proof of the L' Hospital's Rule is not required.

Detailed Content	Time Ratio	Notes on Teaching
4.2 Rate of change	3	The meaning of $\frac{dy}{dx}$ as the rate of change of y with respect to x should be introduced and thoroughly discussed with reference to some common quantities like velocity and acceleration etc. Examples for consideration: (1) A snowball is melting with its volume decreasing at a constant rate of x cm^3/s. When its radius is a cm, find (a) the rate of change of its radius; (b) the rate of change of the surface area. (2) The displacement x of a moving particle measured from a fixed point at time t is given by $x = a\sin t + b\cos t$. (a) Find its velocity and acceleration at time t and describe the motion of the particle. (b) Show that the velocity at time t can be expressed as $\sqrt{a^2+b^2-x^2}$.
4.3 Monotonic functions	2	To begin with, teachers may state an intuitively obvious result that if $f'(a) > 0$ then $f(x) < f(a)$ for values of x less than a but sufficiently close to a, and $f(x) > f(a)$ for values of x greater than a but sufficiently close to a. From a geometrical point of view, the result can easily lead to the statement that f(x) is strictly increasing at x = a. (N.B. It is assumed that the function under consideration is continuous and differentiable.) Similar description should be provided for f(x) strictly decreasing at x = a. Following this, the idea of monotonic increasing may be presented as follows: if $f'(x) > 0$ for every x of the interval, then f(x) is a monotonic increasing function throughout the interval. Teachers should help the students to derive the following important result: if $f'(x) > 0$ throughout the interval, (a, b), f(x) is continuous at x = a and $f(a) \geq 0$, then f(x) is positive throughout the interval. A parallel treatment for monotonic decreasing function is expected and this may be done by the students.

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Detailed Content	Time Ratio	Notes on Teaching
		This result is of special relevance in proving inequalities like (i) $(1 + x)^\alpha \leq 1 + \alpha x$ for $0 < \alpha < 1$ and $x \geq -1$ (ii) $(1 + x)^\alpha \geq 1 + \alpha x$ for $\alpha < 0$ or $\alpha > 1$ and $x \geq -1$ (iii) $\sin x > x - \frac{x^3}{3!}$ for $x > 0$
4.4 Maxima and minima	5	The geometrical interpretation of derivative as the gradient of a curve should be explained and emphasized following the introduction of the definition of the gradient of a curve. In this connection, the visualization of a curve being increasing or decreasing can be once again reinforced. Consequenting upon the mastery of this knowledge, students may then be led to acquire the ability of identifying points of local maximum and local minimum (i.e. the turning points of the curve.) They should be helped to appreciate the conditions for the occurrence of local extrema, like the following version: For a function f(x) (a) find a such that $f'(a) = 0$ and (b) test the sign of $f''(a)$ or test for change of sign of $f'(x)$ in a neighbourhood of a. Teachers should remind students of the following points: (i) Local or relative extrema are not necessarily the global or absolute extrema; (ii) turning points may occur at points where the derivatives do not exist; e.g. $y = x^{2/3}$ and $y = \|x\|$; (iii) Stationary points are points whose derivatives are zero; (iv) $f'(a) = 0$ is NOT sufficient to conclude that at x = a there is a local extremum; e.g. $x^3 \sin(\frac{1}{x})$ and x^3. Examples illustrating the foregoing skills and remarks should be worked out and discussed thoroughly with the students prior to the discussion on point of inflection. The procedures commonly adopted is as follows: (a) find a such that $f''(a) = 0$ (b) test $f''(x)$ for change of sign in a neighbourhood of a.

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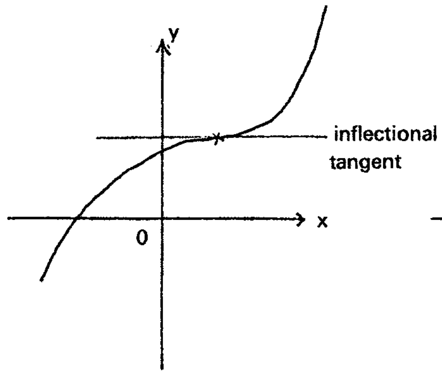
Detailed Content	Time Ratio	Notes on Teaching

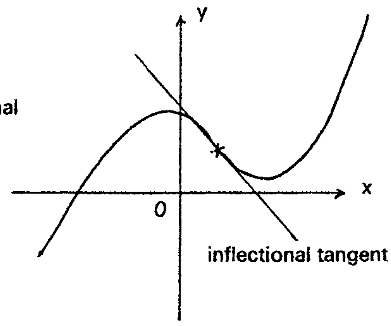
Teachers should remind students of the following points:

(i) at points of inflection, the derivative may not be equal to zero;

(ii) $f''(a) = 0$ is not sufficient to conclude the occurrence of an inflection point at x = a, e.g. x^4.

Hence, diagrams showing the different orientation of the inflectional tangents are very helpful.





As final touching up, teachers may elaborate briefly on the idea of absolute extrema in relation to the domain of the function being extended or shrinked.

Detailed Content	Time Ratio
4.5 Curve Sketching	6

Prior to full embarkation on this topic, students should be taught how to determine the vertical, horizontal and oblique asymptotes to a curve whenever they exist. It is recommended that illustrating examples should go with the explanation. For example, the curve of $y = \frac{x^3}{x^2 - 1}$.

Students should be alerted to look for points of discontinuity where vertical asymptotes are likely to exist. They should also be led to study the behaviour of the function at infinity, viz,

Detailed Content	Time Ratio	Notes on Teaching

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} \to x$ as x is sufficiently large and so they can realize that y = x represents an asymptote.

To accomplish this topic, teachers are advised to help students look for, extract and organize every single bit of glue and information so as to sketch the curve in a more systematic way. The following points are noteworthy:

(1) Symmetry about the axes: inspect the equation to detect any symmetry using rules

(a) if no odd powers of y appear the curve is symmetrical about the x-axis.

(b) if no odd powers *of x* appear the curve is symmetrical about the y-axis.

e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric about both axes.

(2) Limitation on the range *of* values of x and y.

e.g. (*a*) For $y^2 = 4x$, x must be non-negative whi le all values of y are permissible.

(*b*) For $x^2y^2 = a^2(x^2 - y^2)$, upon re-writing $y^2 = \frac{a^2x^2}{x^2 + a^2}$, thus all values *of* x are permissible whereas upon another presentation as $x^2 = \frac{a^2y^2}{a^2 - y^2}$, it is obvious that |y| < a. Actually, the curve is included between the asymptotes y = ± a.

(3) Intercepts with the axes or any obvious points on the curve.

e.g. For $y = \frac{x(x+2)}{x-2}$, the curve intercepts the x-axis at −2 and 0 and there is no intercept made with the y-axis except at the origin.

(4) Points of maximum, minimum and inflection.

(5) Asympotoes to the curve

To encompass the various facets, examples should be worked out for students' heeding, however for trigonometric functions, the attention to the period of the curve is desirable. Regarding curve given by parametric equations, no specific rules can be taken heed of and it is advisable to obtain the corresponding Cartesian representation prior to sketching it.

Some typical curves illustrating the above steps should be sketched for students' reference. The following may be considered:

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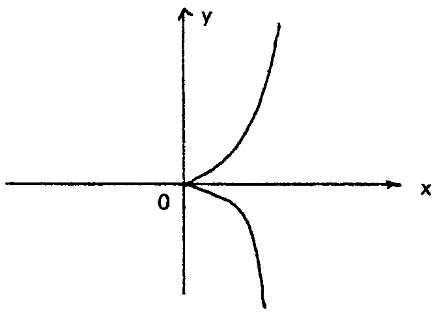
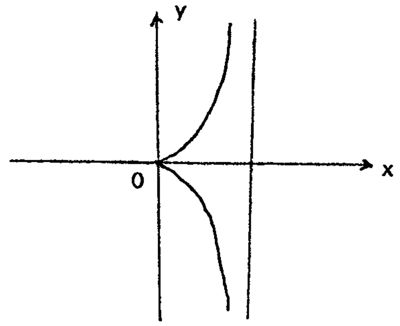
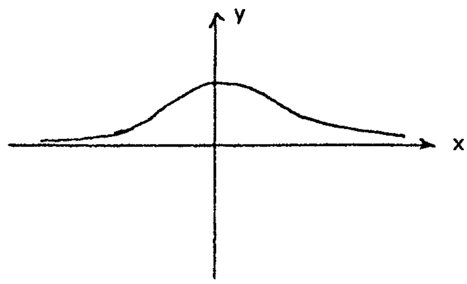
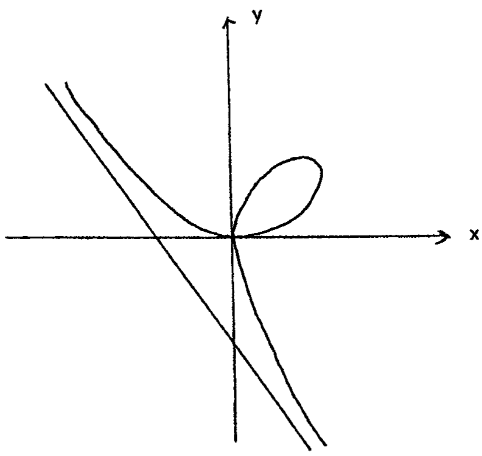
Detailed Content	Time Ratio	Notes on Teaching
		1. $y^2 = ax^3$  2. $y^2 (a - x) = x^3$ 

Detailed Content	Time Ratio	Notes on Teaching
		3. $y = \frac{1}{x^2 + a^2}$  4. $x^3 - 3axy + y^3 = 0$ 

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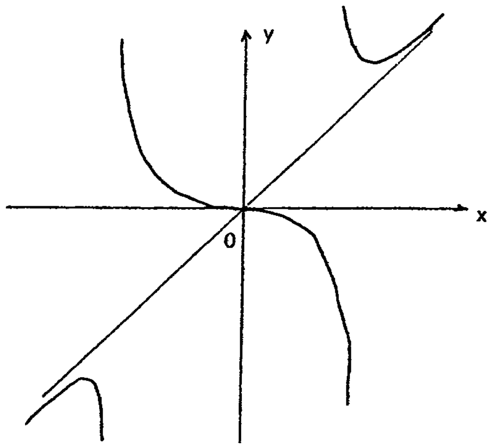
Detailed Content	Time Ratio	Notes on Teaching
		5. $y = \dfrac{x^3}{x^2 - 1}$ 
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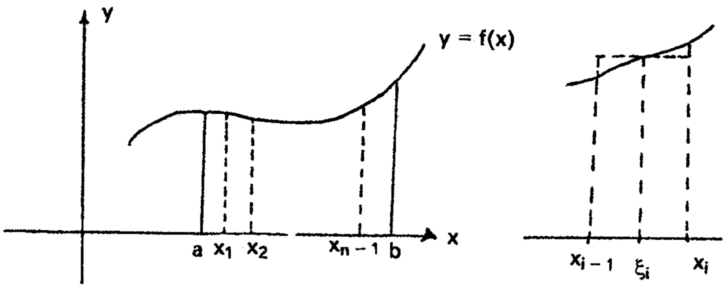
Unit B5: Integration

Objective: (1) To understand the notion of integral as limit of a sum.
(2) To learn some properties of integrals.
(3) To understand the Fundamental Theorem of Integral Calculus.
(4) To apply the Fundamental Theorem of Integral Calculus in the evaluation of integrals.
(5) To learn the methods of integration.
(6) To acquire the first notion of improper integral.

Detailed Content	Time Ratio	Notes on Teaching
5.1 The Riemann definition of integration	5	The theory of the definite integral can be presented in two distinct ways, according as we adopt the geometrical approach or the analytical approach. In the former, the idea of area is presumed, while in the latter the notion of the definite integral as the limit of an algebraic sum without any appeal to geometry is employed. Teachers should determine their choices and sequences of teaching according to the needs of their students. Teachers may start with a function $f(x) \geq 0$ for easy understanding and the following simplified version of an intuitive approach is for reference: Let the function $f(x) \geq 0$ in the interval $[a, b]$ and therein let the graph of $y = f(x)$ be finite and continuous. 

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Detailed Content	Time Ratio	Notes on Teaching

Partition [a, b] into n subintervals by points x_0, x_1, x_2,…, x_n such that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ and let Δx_i denotes $x_i - x_{i-1}$ and ξ_i be an arbitrary point in $[x_{i-1}, x_i]$. The area of the region bounded by the curve y = f(x), the ordinates x = a and x = b and the x-axis can be approximated by the sum

$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$. Moreover, when n increases and max $(\Delta x_i) \to 0$,

the value of area can be found and such limit of sum is defined as the definite integral of f(x) from x = a to x = b and it is denoted by

$$\int_a^b f(x)dx \quad \text{i.e.} \quad \int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \max(\Delta x_i)\to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

In the notation,

f(x) is called the integrand; a is called the lower limit; b is called the upper limit and the sum is called the Riemann sum.

Teachers should then generalize the discussion to obtain the definition of Riemann sum of a general function f(x).

During the discussion with students, the following points should be highlighted:

(1) The partition of [a, b] into subintervals is arbitrary;

(2) $\xi_i \in [x_{i-1}, x_i]$ is arbitrary

(3) The definition of the definite integral as the limit of sum presupposes that a < b. Its value when a > b is defined by

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$ and when a = b by

$$\int_a^a f(x)\,dx = 0$$ (N.B. These results may become theorems if definite integrals are defined by means of the function F(x); for

F(b) – F(a) = –[F(a) – F(b)]

F(a) – F(a) =0. Illustrating examples embellishing the verbal presentation of the teachers should be worked out to help students substantiate their understanding. The following examples are for reference.

Detailed Content	Time Ratio	Notes on Teaching

Example 1:

$$\int_a^b e^x\,dx$$

Consider equal intervals $\Delta x_i = \frac{b-a}{n} = h$ (say), then x_0 = a, x_1 = a + h, …, x_{i-1} = $a + (i-1)h$. Choose ξ_i be x_{i-1} i.e. $\xi_i = a + (i-1)h$. As max $\Delta x_i = \Delta x_i = h$,

$$\int_a^b e^x \, dx = \lim_{h\to0}\sum_{i=1}^{n} e^{\xi_i}h = \lim_{h\to0} h\sum_{1}^{n} e^{a+(i-1)h} = \lim_{h\to0} he^a\sum_{1}^{n} e^{(i-1)h}$$

$$= \lim_{h\to0} he^a\frac{(e^{nh}-1)}{(e^h-1)} = \lim_{h\to0} he^a\frac{(e^{b-a}-1)}{(e^h-1)} = \lim_{h\to0} h\frac{(e^b-e^a)}{(e^h-1)}$$

$$= (e^b - e^a)\lim_{h\to0}\frac{h}{(e^h-1)} = (e^b-e^a)\lim_{h\to0}\frac{1}{e^h} = e^b - e^a .$$

Example 2:

$$\int_a^b x^m\,dx , m \neq -1.$$

Consider n intervals such that $x_0 = a$, $x_1 = ar$,…, $x_i = ar^i$, $x_n = ar^n = b$. When $n \to \infty$, we have $b = ar^n \Leftrightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$ so that $r \to 1$, and max $\Delta x_i = \Delta x_n = x_n - x_{n-1} = ar^n - ar^{n-1} = ar^n(1-r^{-1}) = b(1-r^{-1}) \to 0$

Choose $\xi_i = x_{i-1} = ar^{i-1}$

$$\int_a^b x^m dx = \lim_{r\to1}\sum_{i=1}^{n}(ar^{i-1})^m(ar^i - ar^{i-1})$$

$$= \lim_{r\to1}\sum_{i=1}^{n} a^{m+1}r^{(m+1)(i-1)}(r-1)$$

$$= \lim_{r\to1} a^{m+1}(r-1)\cdot\frac{r^{(m+1)(n)}-1}{r^{m+1}-1}$$

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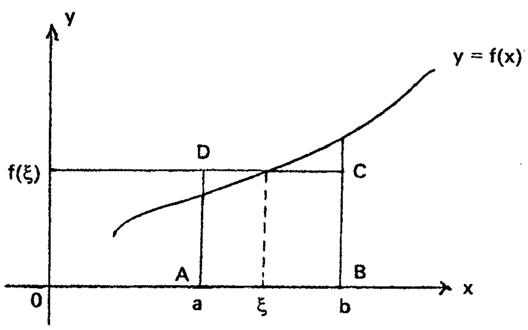
Detailed Content	Time Ratio	Notes on Teaching
		$= \lim_{r\to1} a^{m+1}(r^{(m+1)n}-1)\cdot\frac{r-1}{r^{m+1}-1}$ $= \lim_{r\to1}(b^{m+1}-a^{m+1})\cdot\frac{r-1}{r^{m+1}-1}$ $= (b^{m+1}-a^{m+1})\cdot\lim_{r\to1}\frac{1}{(m+1)r^m}$ $= \frac{b^{m+1}-a^{m+1}}{m+1}$ Last but not least, teachers should also elaborate on (i) If f(x) is continuous on [a, b], then f(x) is integrable over [a, b] (ii) If f(x) is bounded and monotonic in [a, b], then f(x) is integrable over [a, b].
5.2 Simple properties of definite integrals	4	Teachers may help their students derive the following results from the definition. (1) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, k being a constant (2) $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x)+g(x)]\,dx$ (3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any point inside or outside the interval [a, b]. (4) If f(x) ≥ g(x) for all values of x in [a, b], then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ (5) If $\lvert f(x)\rvert \le \phi(x)$ for all values of x in [a, b], then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le \int_a^b \phi(x)\,dx$. In particular, (a) if $\phi(x) = \lvert f(x)\rvert$, then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le \int_a^b \lvert f(x)\rvert\,dx$ (b) if $\phi(x) = M$, M being constant, then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le M(b-a)$.

Detailed Content	Time Ratio	Notes on Teaching
		Simple and straightforward applications like the following may be discussed with the students: (1) If f(x) is positive and monotonic increasing for x > 0, prove that $f(n-1) \le \int_{n-1}^{n} f(x)\,dx \le f(n)$ (2) $\left\lvert\frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx\right\rvert \le \frac{\pi}{4n}$
5.3 The Mean Value Theorem for Integrals	2	A simplified version of the theorem is advisable, viz If f(x) is continuous on [a, b], then there exists a number ξ in (a,b) such that $\int_a^b f(x)\,dx = f(\xi)\cdot(b-a)$ The idea conveyed can easily be visualized through the accompanying diagram. Students should find no difficulty to understand the intrinsic meaning of f(ξ) (b − a) being the area of the rectangle ABCD. 

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Detailed Content	Time Ratio	Notes on Teaching
		If a more formal proof is desirable, it can be furnished by using the properties mentioned in 5.2 together with the properties of continuous function and in particular, the Intermediate Value Theorem.
5.4 Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals	4	The First Fundamental Theorem of Integral Calculus, viz Let f(x) be continuous on [a, b] and let F(x) be defined by $F(x)=\int_a^x f(t)\,dt, a \le x \le b$ then (i) F(x) is continuous in [a, b] (ii) F(x) is differentiable in (a, b) and $\frac{d}{dx}F(x)=f(x)$ or the simplified version If f(x) is continuous, then the function $F(x)=\int_a^x f(t)\,dt$ is differentiable and its derivative is equal to the value of the integrand at the upper limit of integration i.e. F'(x) = f(x). This should be discussed thoroughly with the students and students may be, under the supervision of their teachers, led to prove the Theorem using the Mean Value Theorem for Integral Calculus. (N.B. Teachers should, immediately following this theorem, elaborate on the results follow: (1) the function F(x) whose derivative is equal to the integrand f(x) is called a primitive of f(x). (2) for two such primitives F(x) and G(x) of the same integrand, the derivative of F(x) − G(x) is identically zero, so F(x) − G(x) is constant.) Regarding The Second Fundamental Theorem of Integral Calculus, teachers may again assist their students in the derivation. The version that follows is for consideration: Let f(x), and F(x) be continuous in [a, b]; if $\frac{d}{dx}F(x)=f(x)$ for $a < x < b$, then for $a < x \le b$, $\int_a^x f(t)\,dt = F(x)-F(a)$ and, in particular $\int_a^b f(x)\,dx = F(b)-F(a)$.

Detailed Content	Time Ratio	Notes on Teaching
		Some enlightening examples in evaluating definite integrals by taking it as an infinite sum in the first place and then by finding its primitive as an alternative solution should be worked out so that students' overall understanding on the theorems taught can be strengthened and hence their awareness of the alternative approach in evaluating integrals through the reverse process of differentiation can be promoted. Teachers may start with simpler ones like the following $\int_a^b x^2\,dx = \frac{b^3}{3}-\frac{a^3}{3}$ and end up with other interesting applications like (1) By considering $f(x)=\frac{1}{x}$ in interval [1,2], the result that $\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n} \to$ ($\ell n\,2$ as $n\to\infty$) can be established. (2) By considering $f(x)=\frac{1}{1+x^2}$ over (0,1), one can show that as $n\to\infty$, $n\sum_{r=2}^{n}\frac{1}{r^2+n^2}=\frac{\pi}{4}$.
5.5 Indefinite integration	6	As a continuation, this section is devoted to focus students' attention to the mechanical process of finding .primitive as an alternative approach to evaluate definite integrals. The notation $\int f(x)dx$ representing the indefinite integral of f(x) should be introduced in the sense that If $\frac{d}{dx}F(x)=f(x)$ holds, then F(x) is said to be an Indefinite Integral of f(x) and is denoted by $F(x)=\int f(x)dx$. Teachers should also point out that indefinite integral of f(x) is not unique and that if F(x) is an indefinite integral of f(x), then F(x) +c where c is a constant, is another, treating $\int f(x)dx$ as a primitive of f(x). Students are expected to be able to apply the following formulae for evaluating indefinite integrals. As a matter of fact, they can be encouraged to derive some or all of them.

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Detailed Content	Time Ratio	Notes on Teaching
		(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ (2) $\int \frac{dx}{x} = \ell n(x) + c$ (3) $\int e^x dx = e^x + c$ (4) $\int a^x dx = a^x \ell n a + c$ (5) $\int \sin x \, dx = -\cos x + c$ (6) $\int \cos x \, dx = \sin x + c$ (7) $\int \sec x \tan x \; dx = \sec x + c$ (8) $\int \sec^2 x \, dx = \tan x + c$ (9) $\int \operatorname{cosec} x \cot x \; dx = -\operatorname{cosec} x + c$ (10) $\int \operatorname{cosec}^2 x \; dx = -\cot x + c$ (11) $\int \tan x \; dx = \ell n \lvert \sec x \rvert + c$ (12) $\int \cot x \; dx = \ell n \lvert \sin x \rvert + c$ (13) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ (14) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ Teachers may also remind the students of the following properties (1) $\int kf(x) \, dx = k \int f(x) \; dx$, k is a constant. (2) $\int [f(x) + g(x)] \, dx = \int f(x) \; dx + \int g(x) \, dx$ Students should be encouraged to have adequate practices on a sufficient variety of indefinite integrals in order to testify their mastery of the elementary manipulation to facilitate smoother acquisition of the forthcoming techniques.

Detailed Content	Time Ratio	Notes on Teaching
5.6 Method of integration (A) Method of Substitution	8	It is suggested that the substitution formula $\int f(u) \, du = \int f[g(x)] \, g'(x) \, dx$ need not be proved rigorously, however teachers are advised to start with simpler and obvious ones like $\int \frac{dx}{x+1}, \int (x+1)^{10} \, dx, \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$, $\int \sin^5 x \cos x \, dx, \int \frac{dx}{x \, \ell n x}$ etc. In some integrals when g'(x) does not readily appear, g(x) has to be guessed such as the cases $\int \frac{1+e^x}{1-e^x} dx$, $\int \sqrt{1-x^2} \, dx$ etc, students have to develop the technique through a lot of relevant practices like the following (1) $\int_0^{\pi/2} \frac{dx}{2+\sin x}$ (2) $\int \frac{dx}{\cot x + \operatorname{cosec} x}$ (3) $\int \frac{dx}{\sqrt{e^x - 1}}$ (let $u = e^x$) (4) $\int \frac{e^{2x} dx}{\sqrt[4]{e^x + 1}}$ (let $u = e^x + 1$) (5) $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$ where $b > a$ (let $x = a\cos^2\theta + b\sin^2\theta$) (6) $\int_0^1 \frac{\ell n(1+x)}{1+x^2} dx$ (let $x = \tan\theta$) The following useful results should also be discussed with students with supporting examples for illustration: (1) $\int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx$ and, in particular $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$

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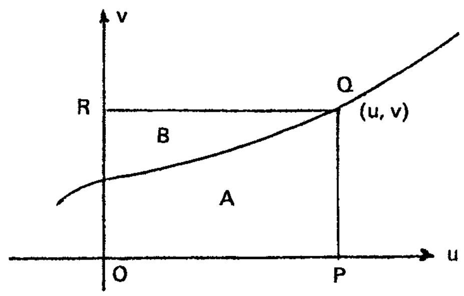
Detailed Content	Time Ratio	Notes on Teaching
		(2) If $f(x)=f(a-x)$, then $\int_0^a xf(x)\,dx=\frac{a}{2}\int_0^a f(x)\,dx$ and in particular $\int_0^{\pi} xf(\sin x)\,dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ (3) If f(x) is periodic with period w then $\int_a^{a+w} f(x)\,dx=\int_0^w f(x)\,dx$ (4) If f(x) is an even function, then $\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ (5) If f(x) is an odd function, then $\int_{-a}^{a} f(x)\,dx=0$ (6) $\int_0^{\pi/2} f(\cos x)\,dx=\int_0^{\pi/2} f(\sin x)\,dx=\frac{1}{2}\int_0^{\pi} f(\sin x)\,dx$ Related examples suggested for consideration are as follows: (1) $\int_0^{\pi}\frac{\cos^3 x}{\sin x+\cos x}\,dx$ (2) $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx$ (3) Show that $\int_0^a x^m(a-x)^n dx=\int_0^a x^n(a-x)^m dx$, and hence evaluate $\int_0^8 x^2\sqrt[3]{8-x}\,dx$ (4) $\int_{-\pi}^{\pi} x^4\sin x dx$ (5) Show that $\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx$ and hence evaluate the integral. (6) Show that $\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}\,dx=\pi/4$.

Detailed Content	Time Ratio	Notes on Teaching
(B) Integration by Parts	3	The integration by parts formula $\int f(x)\,g'(x)\,dx=f(x)\,g(x)-\int g(x)\,f'(x)\,dx$ or $\int udv=uv-\int vdu$ can readily be proved using the intuitive geometrical approach, like  The diagram suggests an informal geometrical interpretation of the formula: Area of region A can be represented by $\int vdu$; Area of region B by $\int udv$; Area of OPQR by uv and hence the formula is readily depicted. Typical examples for illustration include $\int xe^x dx$, $\int x\sin x dx$ and $\int \ell n\, x\, dx$ With the combination of the method of substitution and integration by parts formula, students are able to handle many different kinds of integrals like (1) $\int e^{ax}\cos bx\,dx$ (2) $\int \tan^{-1}x\,\ell n\,(1+x^2)\,dx$ (3) $\int\left(\frac{1}{x}+\frac{1}{x^2}\right)\ell n\,x\,dx$ (4) $\int_0^1 \sin^{-1} x dx$

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Detailed Content	Time Ratio	Notes on Teaching
		(5) $\int_0^1 x\tan^{-1}x\,dx$ (6) $\int_0^{\pi/2}\frac{x\cos x\sin x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$
(C) Reduction Formula	5	Reduction formula is used to express the integral of the general member of a class of functions in terms of simpler member(s) of the class. The reduction formula is generally obtained by applying the method of integration by parts. It is quite extensively used in the integration of trigonometric functions. Typical examples for consideration are as follows: (1) Let I_n denote $\int_0^{\pi/4}\tan^n x\,dx$, show that $I_n=\frac{1}{n-1}-I_{n-2}, n\geq 2$ hence Evaluate I_4. (2) Let $I_n=\int\frac{dx}{(x^2+a^2)^n}$, obtain a reduction formula for I_n and then evaluate $\int_0^a\frac{dx}{(x^2+a^2)^3}$. (3) If $I_n=\int x^n e^{x^2}dx$, show that $I_n=\frac{1}{2}x^{n-1}e^{x^2}-\frac{1}{2}(n-1)\,I_{n-2}$ for n > 2.
(D) Integration by Partial Fractions	4	Integration of rational algebraic functions may be achieved by splitting the expressions into partial fractions. There are four types of fractions in general: $\int\frac{L\,dx}{ax+b},\int\frac{L\,dx}{(ax+b)^r},\int\frac{Lx+M}{(ax^2+bx+c)}dx$ and $\int\frac{Lx+M\,dx}{(ax^2+bx+c)^r}$ Students should be able to handle the first three types without significant difficulty while for the last type, the application of reduction formula is required. Some examples suggested for discussion are (1) $\int_{-1}^1\sqrt{\frac{x+3}{x+1}}\,dx$ (2) $\int\left(\frac{x}{x^2-3x+2}\right)^2 dx$

Detailed Content	Time Ratio	Notes on Teaching
		(3) Let $I_n=\int\frac{dx}{(1-x^4)^n}$, show that $4nI_{n+1}=(4n-1)I_n+\frac{x}{(1-x^4)^n}, n\geq 1$ and evaluate $\int\frac{x^8}{(1-x^4)^4}dx$ However, it is worthwhile for students to note that some integrals cannot be expressed in elementary terms, like $\int e^{-x^2}dx$, $\int\sin(x^2)\,dx,\int\frac{\sin x}{x}dx$, … etc.
5.7 Improper integrals	4	The first notion of improper integral is to be introduced and students are expected to be able to recognize improper integral of the first type viz, $\lim_{b\to\infty}\int_a^b f(x)\,dx$ or $\lim_{a\to-\infty}\int_a^b f(x)\,dx$ which may be simply denoted by $\int_a^{\infty}f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ and improper integral of the second type, viz when $\lim_{x\to a}f(x)=\infty$ $\int_a^b f(x)\,dx=\lim_{h\to 0^+}\int_{a+h}^b f(x)\,dx$ and when $\lim_{x\to b}f(x)=\infty$, $\int_a^b f(x)\,dx=\lim_{h\to 0^+}\int_a^{b-h}f(x)\,dx$. Typical examples of the first type: (1) $\int_0^{\infty}\frac{dx}{1+x}$ (2) $\int_{-\infty}^1\frac{dx}{x^2}$ Teachers are advised to put forth the example $\int_1^{\infty}\frac{dx}{x}$ and pinpoint that this is not an improper integral as the limit does not exist.

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Detailed Content	Time Ratio	Notes on Teaching
		Typical examples of the second type $\int_0^1 \frac{dx}{\sqrt{x}}$ and $\int_{-1}^1 \frac{dx}{\sqrt{1-x}}$. Likewise, teachers may use $\int_0^1 \frac{dx}{\sqrt{x}}$ as illustration that this again is not an improper integral as the limit does not exist either.
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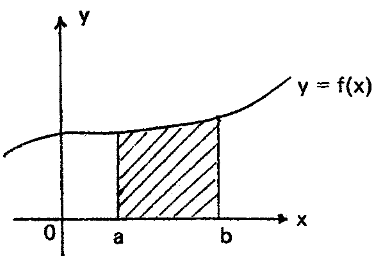
Unit B6: Application of Integration

and

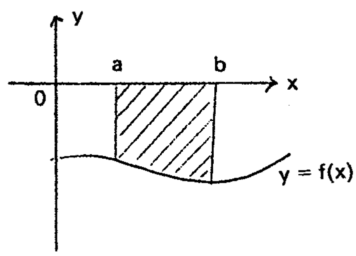
Objective: (1) To learn the application of definite integration in the evaluation of plane area, arc length, volume of solid of revolution and area of surface of revolution.

(2) To apply definite integration to the evaluation of limit of sum.

Detailed Content	Time Ratio	Notes on Teaching
6.1 Plane area	5	As a sequel to the definition of definite integral, the area bounded by a curve y = f(x), the ordinates x = a and x = b and the x-axis can be evaluated in the following ways depending on the nature of the function (being above or below the x-axis): case (i) when y = f(x) is continuous and non-negative in [a, b], then the area so bounded is given by $\int_a^b f(x)\,dx$ case (ii) when f(x) is continuous and non-positive in [a, b], the area is given by $-\int_a^b f(x)\,dx$



case (i)



case (ii)

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Detailed Content	Time Ratio	Notes on Teaching

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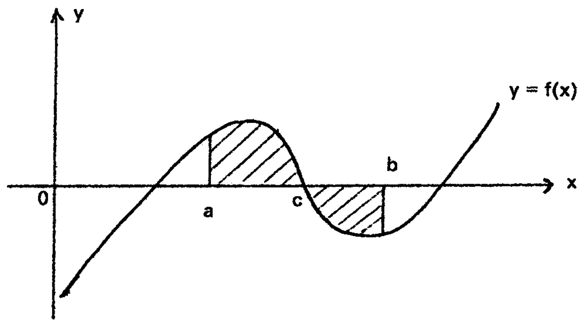
case (iii) when f(x) is continuous and has positive and negative values in [a, b], then the area bounded can be found by following the approach shown in the simplified example that follows.



Area is given by $\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$.

Students should be reminded of the minus sign for area enclosed below the x-axis thus they should be encouraged to have a rough sketch of the function so as to obtain a clearer picture.

Detailed Content	Time Ratio	Notes on Teaching

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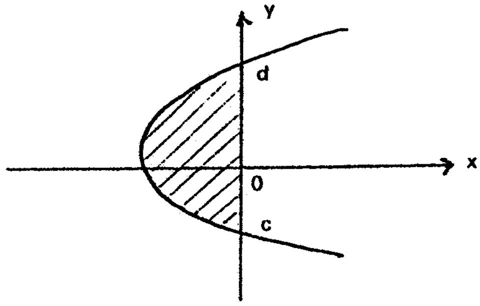
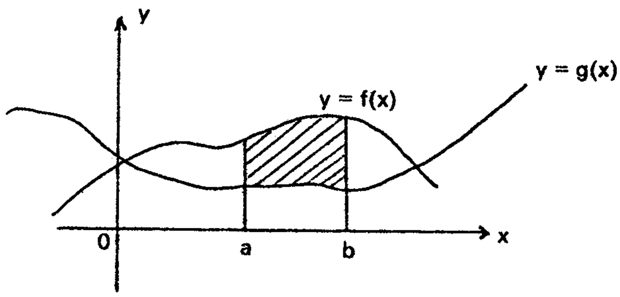
Teachers should also elaborate on the cases where the area enclosed was made with the y-axis like the following diagram.



Area given by $-\int_c^d x\,dy$

For area bounded by two curves, the following approach together with other variations which are illustrated diagrammatically should be discussed thoroughly with students with adequate exemplification.



Area = $\int_a^b [f(x) - g(x)]\,dx$

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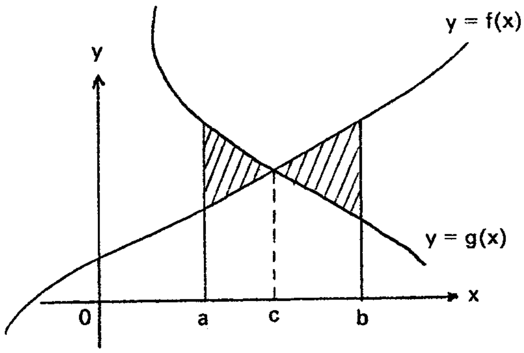
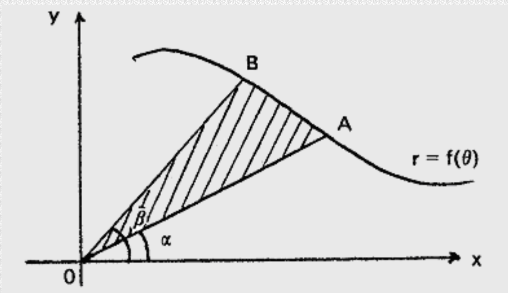
Detailed Content	Time Ratio	Notes on Teaching
		 Area = $\int_a^c [g(x) - f(x)]dx + \int_c^b [f(x) - g(x)]dx$ When the polar equation of the curve is given say, $r = f(\theta)$, then the area bounded by the curved and between the two radii is given by $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$. 

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Detailed Content	Time Ratio	Notes on Teaching
		When the equation of the curve is given in parametric form say, x = x(t); y = y(t), then the area so enclosed is given by $\frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$ where the parameter of A is t_0 and B is t_1 . A good spectrum of worked examples covering some relatively significant variations of the approaches mentioned is highly desirable to enable a better grasp of the skill on the part of the students. The following ones may be considered: (1) Find the area bounded by the parabola $y^2 = 5 - x$ and the line $y = x + 1$. (Note that the area enclosed may be found by integrating with respect to x or to y. Teachers are encouraged to demonstrate both ways.) (2) Show that the area enclosed by the cardioid $r = a(1 + \cos\theta)$ is given by $\frac{3a^2\pi}{2}$. (3) By using the parametric equation $x = a\cos\theta$; $y = b\sin\theta$ of an ellipse, show that the area enclosed is πab. (N.B. Teachers are advised to elaborate on the fact that sometimes the special geometrical properties like symmetry of a figure may help simplify the evaluation process.)
6.2 Arc length	3	The length of arc of the curve y = f(x) between two points on the curve at x = a and x=b is given by $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$ If the curve is given in parametric form x = x(t); y = y(t) then tbe arc length of the curve from $t = t_1$ to $t = t_2$ is given by $\int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$ If the curve is given in polar form $r = r(\theta)$, then the arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ is given by $\int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$.

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Detailed Content	Time Ratio	Notes on Teaching
		The following suggested examples are for discussion and illustration: (1) Show that the perimeter of the closed curve (asteroid) $x = a\cos^3\theta$; $y = a\sin^3\theta$ is 6a. (Symmetry of the curve about the axes helps) (2) Show that the length of 'the circumference of the cardioid $r = a(1 + \cos\theta)$ is 4a. (3) Find the length of the arc of the curve $x^3 = 8y^2$ from x = 1 to x = 3. (N.B. In choosing curves for illustration, teachers should be well aware of the case of an ellipse and, if deem desirable, may lead a brief discussion with students so as to broaden their perspective on other branches of mathematics studies. A brief account in this respect is suggested for reference as follows: For the ellipse $x = a\sin\theta$; $y = b\cos\theta$ $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\cos2\theta + b^2\sin^2\theta = a^2(1-e^2\sin^2\theta)$ where $e = \frac{b}{a}$ (commonly known as the eccentricity). The arc s measured from an extremity of the minor axis is given by $a\int_0^{\phi}\sqrt{1-e^2\sin^2\theta}\,d\theta$. This integral cannot be expressed in terms of elementary functions in a finite form. It is called an elliptic integral of the second kind denoted by $E(e, \phi)$. For the sake of completeness, teachers may also introduce the elliptic integral of the first kind, viz $\int_0^{\phi}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}}$ which is denoted by $F(e, \phi)$.)
6.3 Volume of revolution	4	It is desirable to have some preliminary discussion with the students on the meaning and formation of solids of revolution while the term axis of revolution should also be introduced so that students may be able to identify solids of revolution and visualize the solids formed when certain segment of curve or region is revolving about certain axis. Teachers may then touch upon the two common methods in finding volume of revolution, viz,

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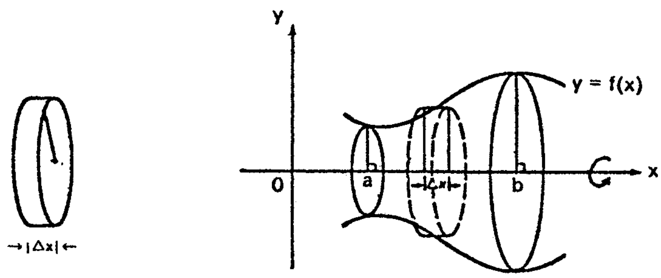
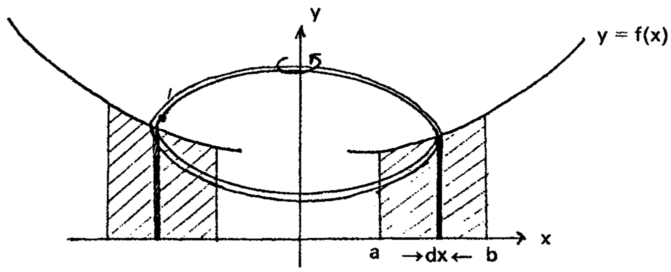
Detailed Content	Time Ratio	Notes on Teaching
		(1) The Disc Method



Teachers should emphasize on the expression of the volume element ΔV or dV that $dV = \pi y^2\,dx$ which is the volume of the disc. The volume of the solid is given by $\pi\int_a^b y^2dx$.

Teachers are also advised to elaborate a bit more on $\pi\int_c^d x^2\,dy$ which is the case when the curve is revolving about the y-axis.

(2) The Shell Method



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Detailed Content	Time Ratio	Notes on Teaching
		In this case the volume element is $2\pi xydx$ and the volume is given by $2\pi\int_a^b xydx$. There are cases in which the solid is formed by revolving about certain lines other than the axes or formed by revolving a region bounded by two curves Thus teachers should elaborate on these cases with the so-called formulae like $\pi\int_a^b [\,[f(x)]^2 - [g(x)]^2]\,dx$ or $\pi\int_c^d [\,[f(y)]^2 - [g(y)]^2]\,dy$ derived for students reference. Adequate illustration is highly recommended. The following are some for reference: (1) Show that the volume generated by rotating the ellipse $x = a\cos\theta; y = b\sin\theta$ about the x-axis is $\frac{4}{3}\pi ab^2$. (N.B. Teachers may request the students to deduce the volume of a sphere of radius r from the given result.) (2) Find the volume of the solid formed by revolving about t h e line x = 2 the region which is Pounded by the curve $y = x^3$, the line x = 2 and the x-axis. (N.B. Teachers are advised to solve this problem using the disc approach as well as the shell approach.)
6.4 Area of surface of revolution	4	The surface area generated by revolving about the x-axis the arc of the curve y = f(x) between x = a and x = b is given by $2\pi\int_a^b yds$ where ds is the element arc length and so the formula usually appears as $2\pi\int_a^b y\sqrt{1+(\frac{dy}{dx})^2}\,dx$ If the said arc length is revolved about the y-axis, the surface area is given by $2\pi\int_c^d xds$ or $2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ where c, and d are respectively the ordinate's of the end points of the arc.

Detailed Content	Time Ratio	Notes on Teaching
		If the curve is given in polar form with 8 as the independent variable, the area generated is given by $2\pi\int_\alpha^\beta y\frac{ds}{d\theta}d\theta$ where $y = r\sin\theta$ and $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$. Similarly, when in parametric form with the curve given by x = x(t); y = y (t) from $t = t_0$ to $t = t_1$, the area of the surface generated is given by $2\pi\int_{t_0}^{t_1} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ The following examples are suggested for class discussion: (1) Show that the area of the surface generated by revolving about the x-axis an arc of the parabola $y^2 = 4x$ between the origin and the point (4, 4) is $\frac{8\pi}{3}(5\sqrt{5}-1)$. (2) Show that the surface area generated by revolving the cardioid r = a(1 + cosθ) about the initial line is $\frac{32}{5}\pi a^2$. (3) Show that the surface area generated by revolving the cycloid x = a(1 − sinθ); y = a(1 − cosθ) about the x-axis is given by $\frac{64}{3}\pi a^2$. (4) Prove that the area of the surface formed by revolving the asteroid $x = a\cos^3 t$; $y = a\sin^3 t$ about the x-axis is $\frac{12}{5}\pi a^2$.
6.5 Limit of Sum	4	In teaching this interesting application of definite integral, it is advisable to start with some simple and obvious series like $\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}+\frac{n}{n^2}$, and students should be provided with adequate hints so that they manage to associate the limit of sum of the series with the limit of sum leading to the relevant definite integral in a suitable interval and with pertinent partition and most important of all with the appropriate integrand, viz $f(x) = x$ in [0, 1] with partition 0, $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, 1.

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Detailed Content	Time Ratio	Notes on Teaching
		It would not be a difficult task for students to establish the result that $\lim_{n\to\infty}(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n}{n^2}) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\cdot\frac{1}{n}$ $= \int_0^1 x\,dx$ $= \frac{1}{2}$ Other examples of great mathematical insight like (1) $\lim_{n\to\infty}(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1}) = \int_0^1 \frac{dx}{1+x} = \ell n\ 2$ (2) $\lim_{n\to\infty}\left(\frac{n}{n^2}+\frac{n}{n^2+1}+\ldots+\frac{n}{n^2+(n-1)^2}\right)$ $= \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$ (3) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n(n+1)}}+\ldots+\frac{1}{\sqrt{n(2n-1)}}\right)$ $= \lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{\sqrt{1+\frac{1}{n}}}+\ldots+\frac{1}{\sqrt{1+\frac{n-1}{n}}}\right)$ $= \int_0^1 \frac{dx}{\sqrt{1+x}} = 2(\sqrt{2}-1)$ may be provided to further their understanding and manipulative technique. The following example is worth discussing as it brings exhilarating result: To find $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$ by transforming it into $\ell n\ y = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\ell n\left(\frac{i}{n}\right)$ where $y = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$ $= \int_0^1 \ell n\ x\ dx = -1$ thus

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Detailed Content	Time Ratio	Notes on Teaching
		obtain $y = e^{-1}$ (N.B. Teachers may relate this part to Unit B5 in which the idea of Riemann sum expressed as a limit of sum of a series is touched upon.) However, students should be reminded that not all limits of sum can be dealt with using this approach, the harmonic series, $\sum\frac{1}{n}$, is an example.
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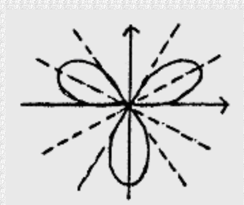
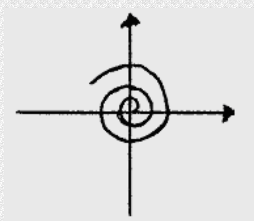
Unit B7: Analytical Geometry

Objective: ~~(1) To learn polar coordinates as another system other than the rectangular coordinate system.~~
(2) To learn the conic sections.
(3) To study locus problems algebraically.
(4) To solve related problems.

Detailed Content	Time Ratio	Notes on Teaching
7.1 Basic knowledge in coordinate geometry	5	Besides the knowledge in their secondary mathematics, students should acquire the following knowledge before they go on to the other topics in this unit: (1) external point of division; (2) area of rectilinear figure using $\frac{1}{2}\times\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$ (3) angle between two lines using $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$; (4) the normal form of a straight line; (5) angle bisectors of two straight lines; (6) family of straight lines and (7) family of circles.
		Students should be able to make conversions between polar and rectangular coordinate systems. They should know how to change the equation of a curve in polar form into the corresponding rectangular form and vice versa. Polar to rectangular: $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ Rectangular to polar: $\begin{cases} r=\sqrt{x^2+y^2} \\ \tan\theta=\frac{y}{x} \end{cases}$ where $x\neq 0$ and θ is determined by the quadrant in which (x, y) lies
7.2 Sketching of curves in the polar coordinate system	4	Students should be able to plot curves with their polar equations given. They are the fundamentals to the topic "Applications of Integration". The following are some suggested simple curves in their polar forms that the students should be able to sketch:

Detailed Content	Time Ratio	Notes on Teaching
		(1) the straight line: θ = k where k is a positive constant; $r\cos\theta$ = a (vertical line). (2) the circle: r = k where k is a positive constant; $r = \sin\theta$. (3) the parabola: $r(1 + \cos\theta)$ = k where k is a positive constant (4) the cardioid: $r = a(1 - \cos\theta)$ where k is a positive constant; (5) the rose curve: $r = a\sin 3\theta$  (6) the spiral: $r = \theta$ 
7.3 Conic sections in rectangular coordinate system	7	Students should be able to distinguish the conic sections in the standard position, namely, $y^2 = 4ax$ (parabola) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (hyperbola) $xy = c^2$ (rectangular hyperbola)

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Detailed Content	Time Ratio	Notes on Teaching
		The parametric representation of the conk sections should also be taught, namely $y = r\sin\theta$; $x = r\cos\theta$ (circle) $y = b\sin\theta$; $x = a\cos\theta$ (ellipse) $y = b\tan\theta$; $x = a\sec\theta$ (hyperbola) $y = \frac{c}{t}$; $x = ct$ (rectangular hyperbola) $y = 2at$; $x = at^2$ (parabola) The knowledge of asymptotes of a hyperbola is expected. The knowledge of the properties of conic sections such as eccentricity, focus and directrix may be taught but need not be emphasized.
7.4 Tangents and normals of conic sections	6	The knowledge of using various methods to find the equations of tangent to a circle is expected, For the simple circle $x^2 + y^2 = a^2$ the equation of tangent to the circle at (x_1, y_1) a point on the circle, is given by $x_1x + y_1y = a^2$ and the normal by $x_1y - y_1x = 0$. The derivation of these basic results should be provided to lead students' thinking and example like the following should also be worked out with due emphasis on the underlying methodology. To find the equation of tangent at (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ (i) using the property that the tangent is always perpendicular to the radius of the circle; (ii) by letting the equation of tangent be $y = mx + k$ and using the fact that the simultaneous equations $\begin{cases} y = mx + k \\ x^2 + y^2 + 2gx + 2fy + c = 0 \end{cases}$ have equal roots It should be noted that the method in (ii) can be applied to the case where (x_1, y_1) is not on the circle. Hence the result that the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) on the circle given by $x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$ can be obtained and upon generalization, the following results can be obtained: (a) Tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) on the curve is $y_1y = 2a(x + x_1)$

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Detailed Content	Time Ratio	Notes on Teaching
		(c) Tangent to the hyperbola $\frac{x^2}{a^2} - \frac{x^2}{b^2} = 1$ at (x_1, y_1) on the curve is $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$ (d) Tangent to the rectangular hyperbola $xy = c^2$ at (x_1, y_1) on the curve is $y_1x + x_1y = 2c^2$ Once the equation of tangent is obtained, students should have no difficulty to obtain the equation of the normal. Furthermore, the corresponding results when the conic sections concerned are presented in parametric form should also be discussed. Teachers may ask the students to do the derivation for themselves: (a) For the parabola $\begin{cases} x = at^2 \\ y = 2at \end{cases}$; the tangent is $y = \frac{x}{t} + at$ (b) For the ellipse $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$; the tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (c) For the hyperbola $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$; the tangent is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ (d) For the hyperbola $\begin{cases} x = ct \\ y = \frac{c}{t} \end{cases}$; the tangent is $x + t^2y = 2ct$. To consolidate students' mastery of the concept as well as the manipulative technique, some basic ideas concerning the chord of contact should be discussed.
7.5 Locus problems in rectangular coordinate system	5	Cases in which a certain set of points satisfying certain constraints and can be represented by equations in the rectangular coordinate system should be studied, e.g. (1) The locus of a movable point with fix distance from a fixed point is a circle. (2) The locus of a movable point which is equidistant from a fixed point and a fixed line is a parabola. (3) The locus of a point on the rim of a circle when the circle is rolled on a straight line represent a cycloid.

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Detailed Content	Time Ratio	Notes on Teaching
7.6 Tangents and normals of plane curves	4	After students have learnt differential calculus, they should be able to apply differentiation to find the equations of tangent and normal of a plane curve in rectangular coordinate plane. Using differentiation formulae and the chain rule, students can find the equations of tangents and normals of curves whose functions are implicitly defined or in parametric form. In this connection, teachers are advised to determine the teaching sequence of this unit, as a prologue to the harder application of differential calculus or as an epilogue of the basic application of differential calculus, according to the ability of the students.
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