It is given $a,b,c \ge 0$ such that $a+b+c=1$. Prove that

(i) $abc(1-c) \le \frac{1}{4}(a+b)^3$,

(ii) for $0 \le c \le \frac{1}{3}$, $\frac{27abc-c^3}{a+b} \le \frac{13}{9}$,

(iii) $\frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} \le \frac{1}{2}$.

Solution

(i) $abc(1-c) \le \frac{1}{4}(a+b)^3 \Leftrightarrow abc \le \frac{(a+b)^2}{4}$

which follows by $c \le 1$ and $ab \le \frac{(a+b)^2}{4}$.

(ii) By AM - GM inequality,

$$\sqrt[3]{abc} \le \frac{a+b+c}{3}$$

$$27abc \le 1$$

$$\frac{27abc-c^3}{a+b} \le \frac{1-c^3}{1-c} = c^2+c+1 \le \frac{1}{9}+\frac{1}{3}+1 = \frac{13}{9}.$$

(iii) By AM - GM inequality,

$$\sqrt{a^3c} \le \frac{a^2+ac}{2}$$

and,

$$\sqrt{b^3c} \le \frac{b^2+bc}{2},$$

therefore,

$$\frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b} \le \frac{2ab+a^2+b^2+c(a+b)}{2(a+b)} = \frac{(a+b)(a+b+c)}{2(a+b)}$$

$$= \frac{a+b+c}{2} = \frac{1}{2}$$

equality for $a=b=c=\frac{1}{3}$.

Alternative Solutions

$$ab+\sqrt{a^3c}+\sqrt{b^3c}$$

$$=ab+a\sqrt{ac}+b\sqrt{bc}$$

$$\le ab+\frac{a(a+c)}{2}+\frac{b(b+c)}{2} \qquad \text{(AM} \ge \text{GM)}$$

$$=ab+\frac{a(1-b)}{2}+\frac{b(1-a)}{2}$$

$$=ab+\frac{a}{2}-\frac{ab}{2}+\frac{b}{2}-\frac{ab}{2}$$

$$=\frac{a+b}{2}$$

$$\Rightarrow \frac{ab+\sqrt{a^3c}+\sqrt{b^3c}}{a+b}\le\frac{1}{2}$$