Let $a^2+b^2+c^2=3$, where $a,b,c\ge 0$.

(a) Prove the followings :

(i) $a+b+c^2\le \frac{7}{2}$,

(ii) $\frac{77}{27}\le a^2+b^2+c^3\le 3\sqrt{3}$.

(b) Prove that $(a^3+b^3)^2\le (a^2+b^2)^3$.

Hence, or otherwise, deduce that

$$a^3+b^3+c^4\le 9$$

$$\left[\begin{array}{l}\text{Hint: For (b), you may use the identity for all real } c,\\ \quad (9-c^4)^2-(3-c^2)^3\equiv (c^2-3)^2(c^2+1)(c^2+6)\end{array}\right]$$

Solution

(a)

(i) $$a+b+c^2=a+b+(3-a^2-b^2)=\frac{7}{2}-\left(a-\frac{1}{2}\right)^2-\left(b-\frac{1}{2}\right)^2\le\frac{7}{2}$$

Equality occurs only when $a=b=\frac{1}{2}$ and $c=\sqrt{\frac{5}{2}}$.

Alternative Solutions (1)

$$\begin{aligned}a+b+c^2&=a(1-a)+b(1-b)+3\\ &\le\left(\frac{a+1-a}{2}\right)^2+\left(\frac{b+1-b}{2}\right)^2+3 \qquad (\text{A.M.}\ge\text{G.M.})\\ &\le\frac{7}{2}\end{aligned}$$

Alternative Solutions (2)

Rewrite it as:

$$a^2+b^2+\frac{1}{2}\ge a+b$$

A.M.-G.M. gives:

$$(a^2+b^2)+\frac{1}{2}\ge\sqrt{2(a^2+b^2)}\ge a+b$$

so it’s done.

(a)

(ii) $a^2+b^2+c^3=c^3-c^2+3$

$\frac{77}{27}\le a^2+b^2+c^3 \Leftrightarrow$

$(3c+1)(3c-2)^2\ge 0$ which is obviously true.

Equality holds $a^2+b^2=\frac{23}{9}$ and $c=\frac{2}{3}$.

$a^2+b^2+c^3\le 3\sqrt{3}\Leftrightarrow$

$c^2(c-1)\le 3\sqrt{3}-3$

Since $a,b,c\ge 0$, $c\le\sqrt{3}$,

When $c<1$,

$c^2(c-1)<0<3\sqrt{3}-3.$

When $c\ge 1$,

$c^2(c-1)\le 3(\sqrt{3}-1)=3\sqrt{3}-3.$ Equality holds $a=b=0$ and $c=\sqrt{3}$.

<u>Alternative Solutions</u>

$a^2+b^2+c^3=3-c^2+c^3, 0\le c\le\sqrt{3}.$

Let $f(x)=3-x^2+x^3$, $f'(x)=-2x+3x^2=x(3x-2)$ for $0\le x\le\sqrt{3}$.

$f(\frac{2}{3})$ is the absolute minimum and $f(\sqrt{3})$ is the absolute maximum.

$f(\frac{2}{3})\le f(c)\le f(\sqrt{3})$

$\frac{77}{27}\le a^2+b^2+c^3\le 3\sqrt{3}$

(b) It is equivalent to proving

$3a^2b^4+3a^4b^2\ge 2a^3b^3$

or,

$3a^2+3b^2\ge 2ab$

and by A.M.-G.M.,

$a^2+b^2\ge 2ab$

so the inequality is true with equality for $a=0, b=0, a=b=0$.

Since $a^3+b^3\le\sqrt{(a^2+b^2)^3}=\sqrt{(3-c^2)^3}$

so we just need to prove that

$(9-c^4)^2\ge(3-c^2)^3$

$\Leftrightarrow (c^2-3)^2(c^2+1)(c^2+6)\ge 0$

clear true. Equality holds $a=b=0$ and $c=\sqrt{3}$.

Alternative Solutions (by bearwing)

$(9-c^4)^2 \geq (3-c^2)^3 \geq (a^3+b^3)^2$ and $9-c^4 \geq 0$ gives

$a^3+b^3+c^4 \leq 9$